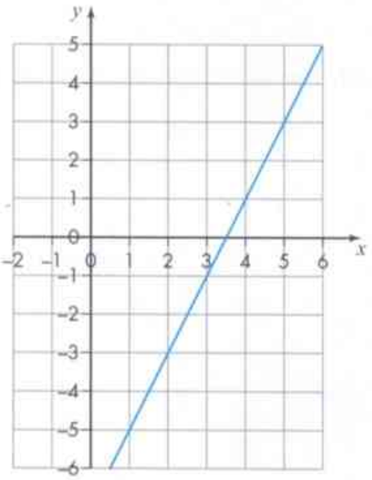
12 This is a graph of $y = 2x - 7$.



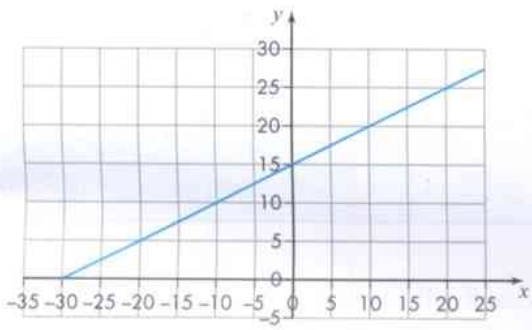
Use the graph to solve these equations.

a $2x - 7 = 5$

b $2x - 7 = 0$

c $2x - 7 = -3$

13 This is a graph of $y = 0.5x + 15$.



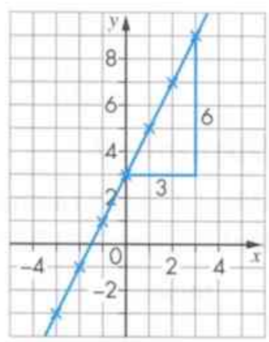
Use the graph to solve these equations.

a $0.5x + 15 = 25$ b $0.5x + 15 = 10$

15.2 The equation $y = mx + c$

Gradient

The **slope** of a line is called its **gradient**. The steeper the slope of the line, the larger the value of the gradient.



The graph shows the line with equation $y = 2x + 3$. You can measure the gradient of the line by drawing a right-angled triangle that has part of the line as its hypotenuse (sloping side). The gradient is then given by:

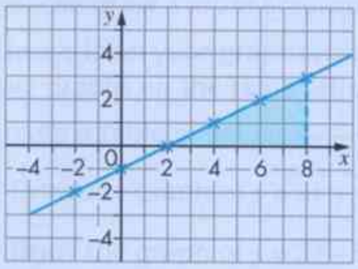
$$\text{gradient} = \frac{\text{distance measured up}}{\text{distance measured along}} = \frac{6}{3} = 2$$

In fact, if the equation of the line is $y = mx + c$, then m is the gradient of the line.

Example 2

Show that the line with the equation $y = 0.5x - 1$ has a gradient of 0.5.

x	−2	0	2	4	6
y	−2	−1	0	1	2



This is a graph of the line.

From the shaded triangle, you can see that the gradient is:

$\frac{3}{6} = 0.5.$

What does the 'c' in $y = mx + c$ represent?

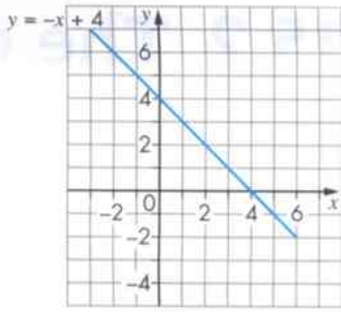
From the graphs above you can see it is the value of y, where the line crosses the y-axis.

$y = 2x + 3$ passes through (0, 3).

$y = 0.5x - 1$ passes through (0, −1).

Note that a line that slopes downwards from left to right has a negative gradient.

This line has a gradient of −1.



Summary

When a graph can be expressed in the form $y = mx + c$, the **coefficient** of x, which is m, is the **gradient**, and the constant term, which is c, is the **intercept** on the y-axis.

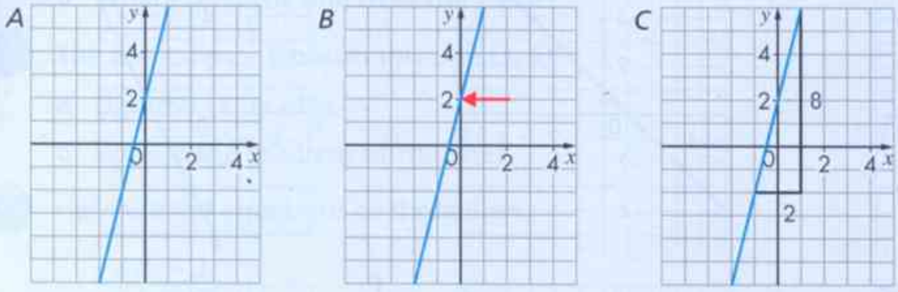
This means that if you know the gradient, m, of a line and its intercept, c, on the y-axis, you can write down the equation of the line immediately.

For example, if $m = 3$ and $c = -5$, the equation of the line is $y = 3x - 5$.

This gives a method of finding the equation of any line drawn on a pair of coordinate axes.

Example 3

Find the equation of the line shown in diagram **A**.



First, find where the graph crosses the y-axis (diagram **B**).

So $c = 2$

Next, measure the gradient of the line (diagram **C**).

y-step = 8

x-step = 2

gradient = 8 ÷ 2 = 4

So $m = 4$

Finally, write down the equation of the line: $y = 4x + 2$.

EXERCISE 15B

1 You drew the graphs of the lines with these equations in Exercise 15A, questions **1–5**.

In each case state the gradient.

Then check from your drawing that you are correct.

a $y = 3x + 4$

b $y = 2x - 5$

c $y = \frac{x}{2} - 3$

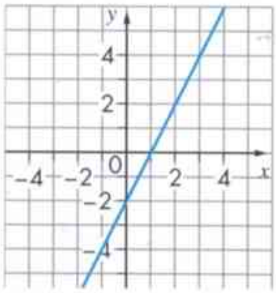
d $y = 3x + 5$

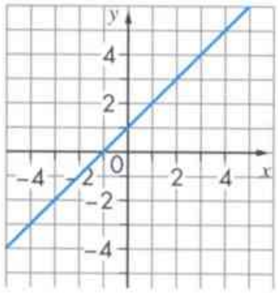
e $y = \frac{x}{3} + 4$

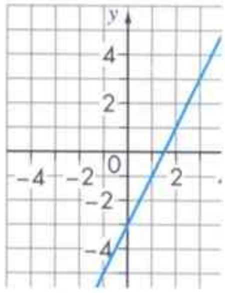
CORE

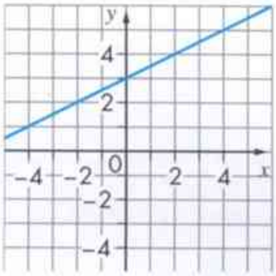
CORE

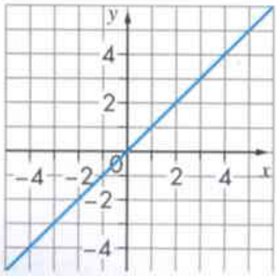
2 Give the equation of each of these lines, all of which have positive gradients. (Each square represents one unit.)

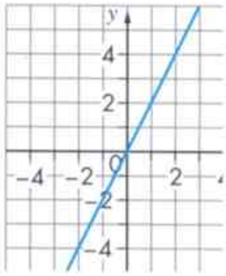
a 

b 

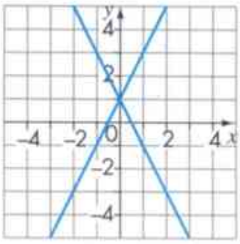
c 

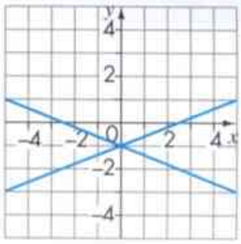
d 

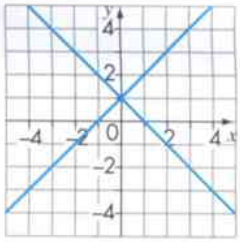
e 

f 

3 In each of these grids, there are two lines. (Each square represents one unit.)

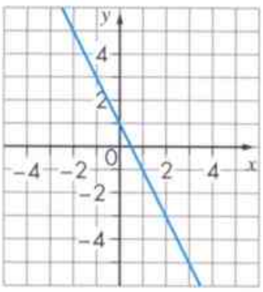
a 

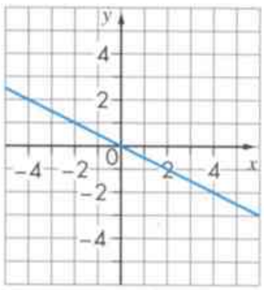
b 

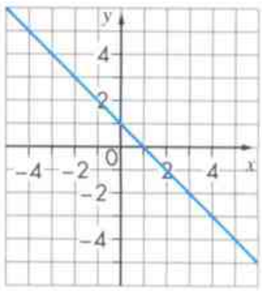
c 

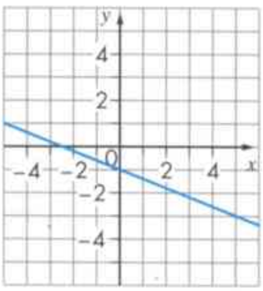
For each grid find the equation of each of the lines.

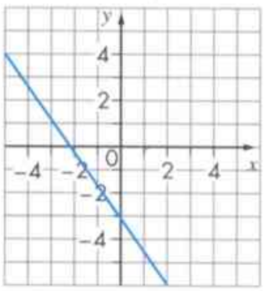
4 Give the equation of each of these lines, all of which have negative gradients. (Each square represents one unit.)

a 

b 

c 

d 

e 

5 The line $y = 4x + c$ passes through (1, 7).

a Find the value of c.

b Where does the line cross the y-axis?

6 The line $y = mx - 6$ passes through (3, 6).

a Find the value of m.

b What is the gradient of the line?

7 Here are the equations of three lines.

$y = 4x - 2$

$y = 3x - 4$

$y = 4x + 5$

Which of these lines are parallel? Explain how you know.

8 Find the equation of a straight line that passes through the point (5, 3) and the origin.

9 a Draw a graph of the line $y = 0.5x + 3$.

b Draw the reflection of the line $y = 0.5x + 3$ in the y-axis.

c Find the equation of the line in part **b**.

d Draw the reflection of the line $y = 0.5x + 3$ in the x-axis.

e Find the equation of the line in part **d**.

15.3 More about straight-line graphs

E

The equation of a straight line is not always written in the form $y = mx + c$

For example, $y = -2x + 3$ is the equation of a straight line with a gradient of –2 and an intercept on the y-axis of (3, 0)

By adding $2x$ to each side the equation can be written as $y + 2x = 3$ or $2x + y = 3$

By subtracting 3 from each side it can be written as $2x + y - 3 = 0$

Here are some more examples.

Equation	Alternatives
$y = -4x - 2$	$y + 4x = -2$ or $4x + y + 2 = 0$
$y = -5x + 1$	$y + 5x = 1$ or $5x + y = 1$
$y = \frac{3}{2}x + 1$	$2y = 3x + 2$
$y = 4x - 12$	$y = 4(x - 3)$

Example 4

The equation of a straight line is $3x + 4y = 24$

a Find the gradient of the line.

b Find the intercept on the y-axis.

c Draw a graph of the line.

a Rewrite the equation.

Subtract $3x$ from both sides: $4y = -3x + 24$

Divide both sides by 4: $y = -\frac{3}{4}x + 6$

The gradient is $-\frac{3}{4}$

b The intercept is (0, 6)

c You know that (0, 6) is on the line.

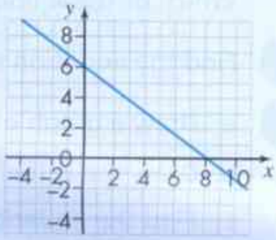
To find another point, substitute $y = 0$ into the original equation.

$3x + 0 = 24$

$3x = 24$

$x = 8$ The point (8, 0) is also on the line.

Draw the line by joining (0, 6) and (8, 0)



EXERCISE 15C

EXTENDED

1 The equation of a straight line is $5x + 2y = 10$

a Write the equation in the form $y = mx + c$

b Find the gradient and the intercept on the y-axis.

c Draw a graph of the line.

2 The equation of a straight line is $x + 3y = 15$

a Write the equation in the form $y = mx + c$

b Find the gradient and the intercept on the y-axis.

c Draw a graph of the line.

3 The equation of a straight line is $x + 2y + 6 = 0$

a Write the equation in the form $y = mx + c$

b Find the gradient and the intercept on the y-axis.

c Draw a graph of the line.

EXTENDED

4 Here are the equations of straight lines.

Write each one in the form $y = mx + c$

a $x + y + 20 = 0$ **b** $y = 3(x + 5)$ **c** $7x + 10y = 30$

d $x - 5y = 40$ **e** $2(y + 1) = 3x$ **f** $12 - 2y = x$

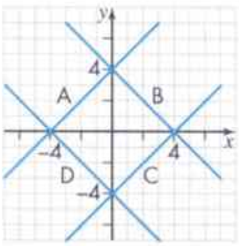
5 Find the gradient and the intercept on the y-axis for each of the lines in question **4**.

6 Find the gradient of each of these straight lines.

a $4x + 4y = 15$ **b** $x = 3y + 6$ **c** $y = 6(x - 5)$

d $5x + 10y = 28$ **e** $10x = 5y + 28$ **f** $28x - 5y = 10$

7



Match the equation to the line.

a $x - y = 4$ **b** $y - x = 4$ **c** $x + y + 4 = 0$ **d** $x + y = 4 = 0$

8 Find the odd one out. Give a reason for your answer.

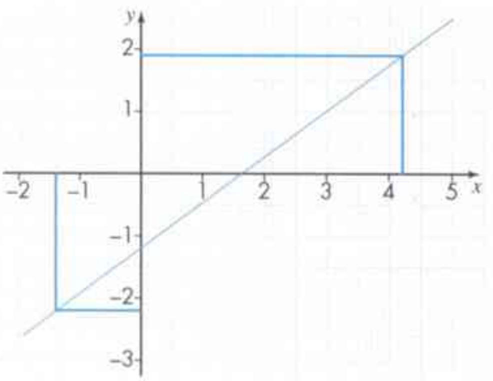
a $2x + 3y = 12$ **b** $y = 4 - \frac{2}{3}x$ **c** $6y = -4x + 24$ **d** $3x - 2y = 12$ **e** $x = 6 - 1.5y$

9 Find the points where each of these straight lines cross the axes.

a $3x + 5y = 45$ **b** $2y = 20 - x$ **c** $12x + 6y + 60 = 0$

15.4 Solving equations graphically

Here is a graph of $y = 0.73x - 1.15$



You can use this graph to solve equations.

Example 5

Use the graph to solve the equations:

a $0.73x - 1.15 = 0$

b $0.73x - 1.15 = -2.2$

c $0.73x - 2.65 = 0$

---

a Since the equation of the line is $y = 0.73x - 1.15$ then $0.73x - 1.15 = 0$ when the y coordinate is zero.

This is where the line crosses the x-axis.

Reading from the graph, $x = 1.6$

Because this is taken from a graph, it can only be correct to 1 d.p.

b To solve the equation $0.73x - 1.15 = -2.2$, find -2.2 go across the line and then up to the x-axis. The lines on the graph show this.

The solution is the x-coordinate. That is $x = 1.6$

c If $0.73x - 3.05 = 0$ then $0.73x - 1.15 - 1.9 = 0$

So $0.73x - 1.15 = 1.9$

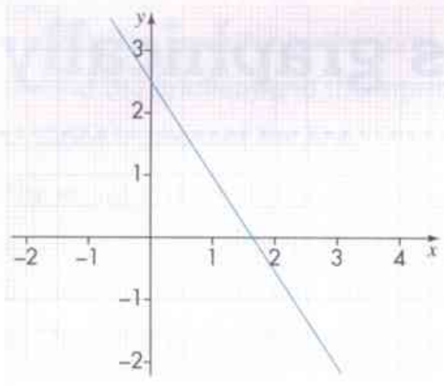
Find 1.9 on the y-axis, go across to the line and down to the x-axis.

The solution is $x = 4.2$

EXERCISE 15D

CORE

1



This is a graph of $y = 2.45 - 1.37x$

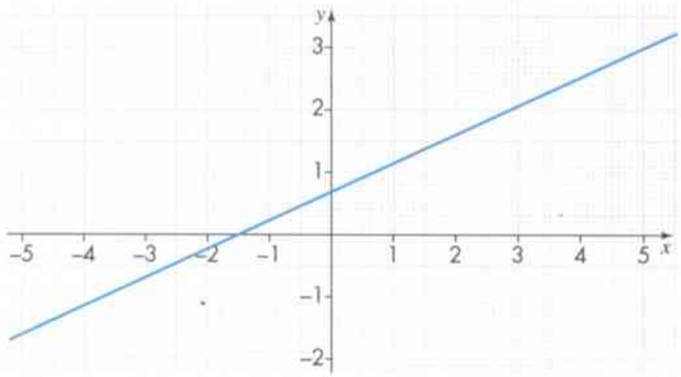
Use it to solve these equations:

a $2.45 - 1.37x = 0$

b $2.45 - 1.37x = 3$

c $2.45 = 1.37x - 1.5$

2

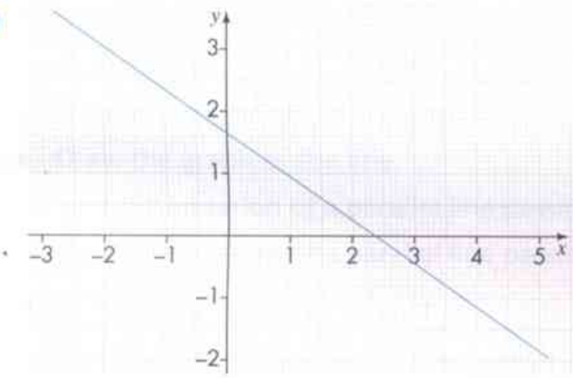


This is a graph of $y = 0.46x + 0.73$

Use the graph to solve these equations:

a $0.46x + 0.73 = 0$ **b** $0.46x + 0.73 = 2.5$ **c** $0.46x + 1.73 = 0$

3

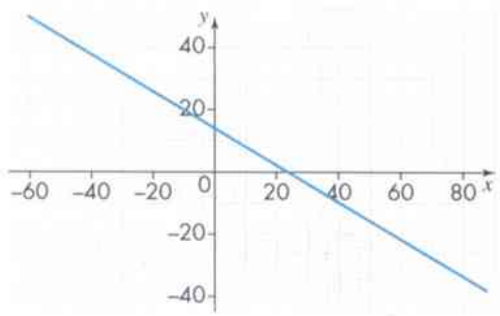


This is a graph of $2.7y + 1.9x = 4.42$

Use the graph to solve these equations:

a $1.9x = 4.42$ **b** $2.7y = 4.42$ **c** $2.7 + 1.9x = 4.42$

4



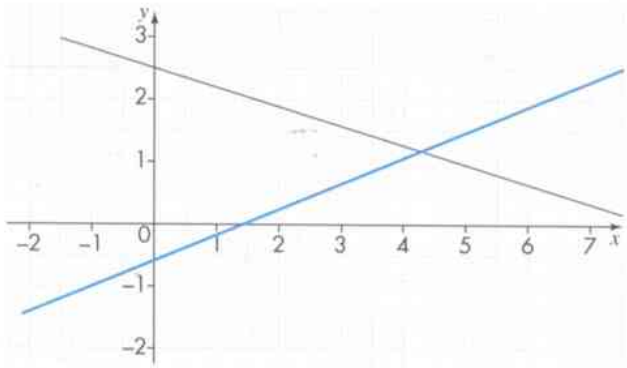
This is a graph of $y = 14.2 - 0.62x$

Use the graph to solve these equations:

a $14.2 - 0.62x = 0$ **b** $14.2 - 0.62x = 32$ **c** $44.2 - 0.62x = 0$

CORE

5



This is a graph of $y = 2.5 - 0.31x$ and $y = 0.42x - 0.6$

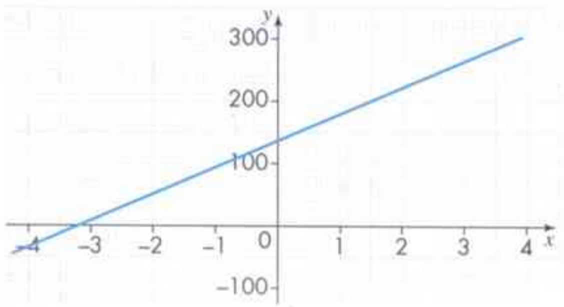
Use the graph to solve these equations:

a $0.42x - 0.6 = 2.3$

b $2.5 - 0.31x = 1.6$

c $0.42x - 0.6 = 2.5 - 0.31x$

6



This is a graph of $y = 43.8x + 136.2$

Use it to solve these equations:

a $43.8x + 36.2 = 0$

b $87.6x + 272.4 = 500$

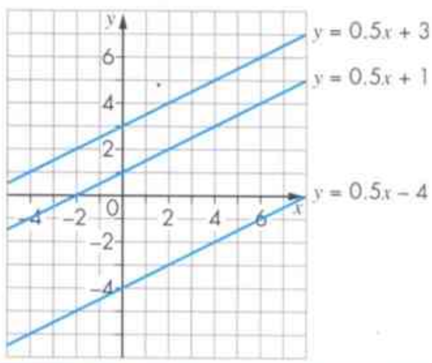
c $438x + 1362 = 1500$

15.5 Parallel lines

Consider the line with the equation $y = 0.5x + 1$.

The gradient is 0.5. Any line **parallel** to it will have the same gradient.

Examples are $y = 0.5x + 3$ and $y = 0.5x - 4$. They all have 0.5 as the coefficient of x.



Example 6

Find the equation of the line parallel to $y = 4x$ that passes through (2, 5).

The line will have the equation $y = 4x + c$ for some value of c.

Substitute the coordinates of (2, 5) into the equation.

$5 = 4 \times 2 + c$

$5 = 8 + c$

$c = -3$

So the equation is $y = 4x - 3$.

EXERCISE 15E

1 A line has the equation $y = 2x + 6$.

- **a** Show that (3, 12) is on this line.
- **b** What is the gradient of the line?
- **c** Draw the graph of the line.
- **d** Find the equation of a parallel line passing through the origin.
- **e** Find the equation of a parallel line passing through (0, 3).

2 A line has the equation $y = \frac{1}{4}x - 1$.

- **a** Find the coordinates of the points where the line crosses the y-axis and the x-axis.
- **b** Find the equation of a parallel line through (8, 5).

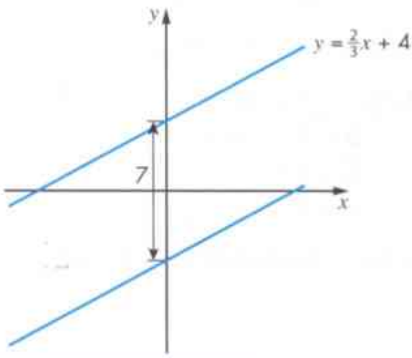
3 A line has the equation $y = 8 - 2x$.

- **a** What is the gradient of this line?
- **b** Where does the line cross the x-axis?
- **c** Draw the line on a grid.
- **d** A parallel line passes through the origin. What is the equation?
- **e** Find the equation of a parallel line through (5, 4).

4
- **a** The line with equation $y = 5x + k$ passes through (2, 11). Find the value of k.
- **b** A line that is parallel to the line in part **a** passes through (4, 11). Find the equation of this line.

5 The two lines in this graph are parallel.

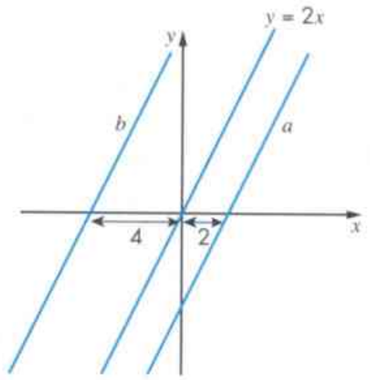
Find the equation of the lower line.



CORE

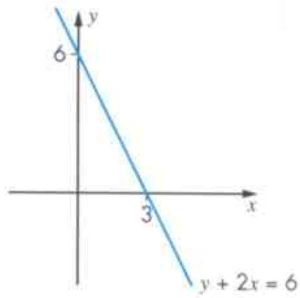
6 The three lines on this graph are parallel.

Find the equations of a and b.



7 The equation of this line can be written as $y + 2x = 6$.

- **a** Show that (1, 4) is on this line.
- **b** Rearrange the formula in the form $y = mx + c$.
- **c** What is the gradient of the line?
- **d** Find the equation of a line that is parallel to this line, but passes through the origin.

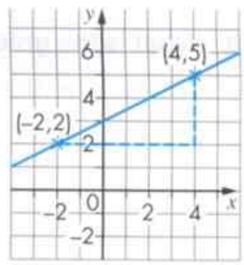


15.6 Points and lines

The equation of a line

If you know the coordinates of two points on a straight line you have enough information to find the equation of the line between them.

Consider the line through (–2, 2) and (4, 5).



Using the triangle shown:

$$\text{gradient} = \frac{\text{difference between } y\text{-coordinates}}{\text{difference between } x\text{-coordinates}}$$

$$= \frac{5-2}{4--2}$$

$$= \frac{3}{6}$$

$$= 0.5$$

So the equation must be $y = 0.5x + c$ for some value of c.

Use the coordinates of either of the points to find c.

Using $x = 4$ and $y = 5$:

$5 = 0.5 \times 4 + c$

$5 = 2 + c$

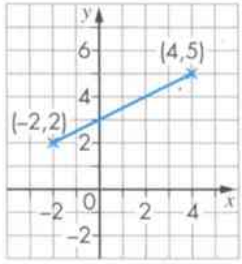
$c = 3$

So the equation is $y = 0.5x + 3$.

Midpoints

The **midpoint** of a **line segment** is the same distance from each end.

To find the coordinates of the midpoint, add the coordinates of the end points and divide by 2.

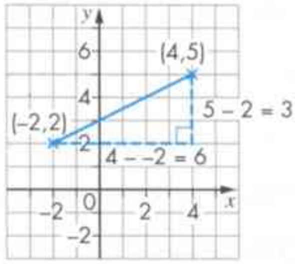


In the example shown:

midpoint is $\left(\frac{-2+4}{2}, \frac{2+5}{2}\right) = (1, 3.5)$

The distance between two points

You can use Pythagoras' theorem (see Chapter 25 section 1) to find the distance between two points.



The lengths of the sides of the right-angled triangle shown are the differences between the x-coordinates and the y-coordinates of the end points.

In this case the lengths are 3 and 6.

Distance between points $= \sqrt{3^2 + 6^2} = \sqrt{45} = 6.71$ to 2 decimal places.

EXERCISE 15F

EXTENDED

1 Find the gradient of the line through each pair of points.

a (4, 0) and (6, 6)

b (0, 3) and (8, 7)

c (2, –2) and (4, 6)

d (1, 5) and (5, 1)

e (–4, 6) and (6, 1)

f (–5, –3) and (4, 3)

Advice and Tips

It will help to plot them on a grid.

2 Find the equation of the line joining each pair of points.

a (0, – 3) and (4, 5)

b (–4, 2) and (2, 5)

c (–1, –6) and (2, 6)

d (1, 5) and (4, –4)

3 Find the midpoints of the line segments joining the points in question **1**.

4 A is (–3, 5), B is (1, 1) and C is (5, 9).

a Draw the triangle ABC on a coordinate grid.

b Find the equation of the straight line through A and C.

c Find the midpoint of AB.

d Find the equation of the straight line through the midpoints of AC and BC.

EXTENDED

5 Find the length of the line segment joining each pair of points.

a (2, 2) and (6, 5) b (–3, 2) and (9, 7)

c (1, 5) and (7, –3) d (–6, –4) and (9, 4)

6 A circle is drawn with its centre at (2, 1) and radius 5.

Show that (–3, 1), (6, 4) and (5, –3) all lie on the circle.

7 Show that the triangle ABC in question **4** is an isosceles triangle.

15.7 Perpendicular lines

E

The lines in this diagram are **perpendicular** to each other.

This means that the angle between them is a right angle, 90°.

You can use the dotted triangles to work out the gradients.

The gradient of line A is $\frac{2}{3}$.

The gradient of line B is $-\frac{3}{2}$ or –1.5.

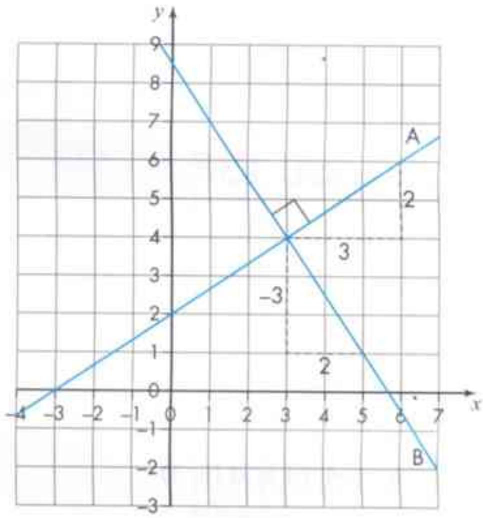
You can see that:

the gradient of line B = –(the reciprocal of line B)

You can also see that the product of the gradients is –1:

$$\frac{2}{3} \times -\frac{3}{2} = -1$$

The gradients of perpendicular lines usually have this property. The only exception is when the gradient of one of the lines is zero.



Example 7

Find the equation of the line that is perpendicular to $2x + y = 3$ and passes through the point (4, 6).

The given equation is equivalent to $y = -2x + 3$.

The gradient is –2.

The gradient of the line perpendicular to this line is $\frac{1}{2}$. (Find the reciprocal, change the sign).

The equation you want is $y = \frac{1}{2}x + c$ where c is a number.

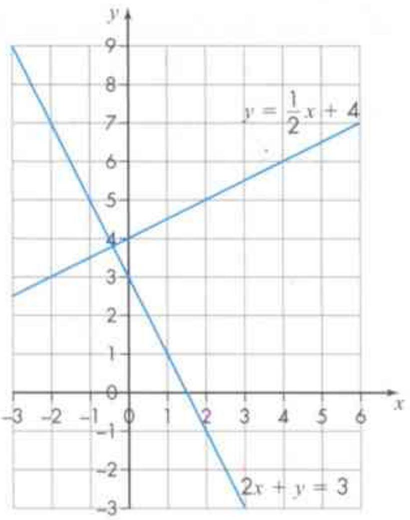
Find the value of c by substituting the coordinates (4, 6) in the equation.

$6 = \frac{1}{2} \times 4 + c$ ÷ $6 = 2 + c$ ÷ $c = 4$

The equation of the perpendicular line is $y = \frac{1}{2}x + 4$.

You can check the answer to example 5 by drawing a diagram.

The lines are perpendicular.



EXERCISE 15G

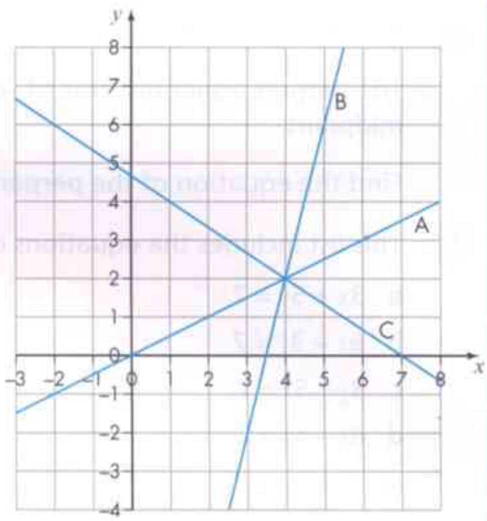
EXTENDED

1 Work out the gradient of a line that is perpendicular to:

a line A

b line B

c line C.



2 Here are the equations of two lines.

$y = 2 + 5x$ $\quad$ $5y + x = 6$

Show that the lines are perpendicular.

3 Work out the gradient of a line that is perpendicular to:

a $y = 3x$ **b** $y = \frac{1}{2}x - 15$ **c** $y = 3.5 - 0.4x$ **d** $y = 7.5x + 5.7$.

4 Work out the gradient of a line that is perpendicular to:

a $x + y + 5 = 0$ **b** $y + \frac{1}{2}x = 4$ **c** $3x + 4y = 5$ **d** $12y - 2x = 15$.

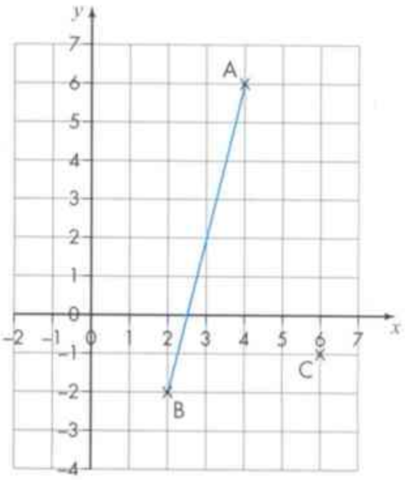
5 Find the equation of the line that is perpendicular to $y = 5x$ and passes through:

a the origin **b** (0, 10) **c** (10, 0).

EXTENDED

6 Find the equation of the line that is perpendicular to $5x + 3y = 30$ and passes through (–20, –10).

7 Find the equation of the line that is perpendicular to AB and passes through C.



8 A is the point (–4, 5) and B is the point (6, 7).

The perpendicular bisector of AB is a line that is perpendicular to AB and passes through its midpoint.

Find the equation of the perpendicular bisector of AB.

9 This list includes the equations of two pairs of perpendicular lines.

a $3x + 5y = 7$

b $6x + 3y = 7$

c $3y - 5x = 7$

d $8x - 4y = 7$

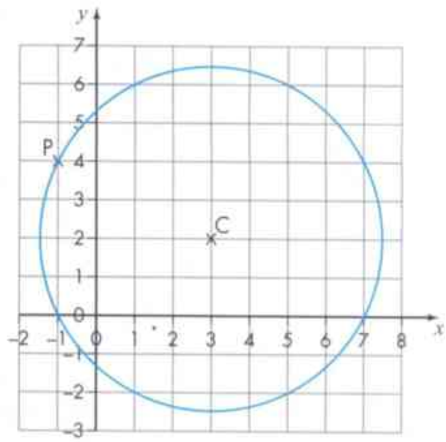
e $4y + 2x = 7$

Which is the odd one out?

10 The centre of a circle is the point C(3, 2).

The circle passes through the point P(–1, 4).

Work out the equation of the tangent to the circle at P.



Check your progress

Core

- I can interpret the equation of a straight-line graph in the form $y = mx + c$
- I can find the equation of a straight line from a graph
- I can find the equation of a straight line parallel to a given line

Extended

- I can interpret and find the equation of a straight-line graph given in different forms
- I can find the gradient of a line parallel to a given line
- I can find the gradient of a line perpendicular to a given line
- I can find the equation of a straight line passing through two points

Chapter 16

Graphs of functions

Topics	Level	Key words
1 Quadratic graphs	CORE	quadratic graph, quadratic equation, parabola
2 Turning points on a quadratic graph	EXTENDED	turning point
3 Reciprocal graphs	CORE	reciprocal
4 More graphs	EXTENDED	cubic, exponential functions, asymptotes
5 Exponential graphs	EXTENDED	exponential growth, exponential decay
6 Estimating gradients	EXTENDED	gradient, tangent

In this chapter you will learn how to:

CORE

- Construct tables of values for functions of the form $\pm x^2 + ax + b$, $\frac{a}{x}$ $(x \neq 0)$ where a and b are integer constants. (C2.11)
- Draw and interpret such graphs. (C2.11)
- Solve quadratic equations approximately by graphical methods. (C2.11)

EXTENDED

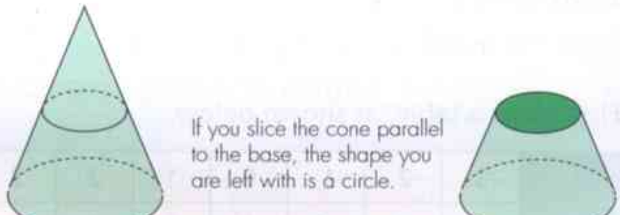
- Construct tables of values and draw graphs for functions of the form ax^n (and simple sums of these) and functions of the form $ab^x + c$ where a and c are rational constants and $n = -2, -1, 0, 1, 2, 3$. (E2.11)
- Solve associated equations approximately by graphical methods. (E2.11)
- Draw and interpret graphs representing exponential growth and decay problems. (E2.11)
- Estimate gradients of curves by drawing tangents. (E2.12)

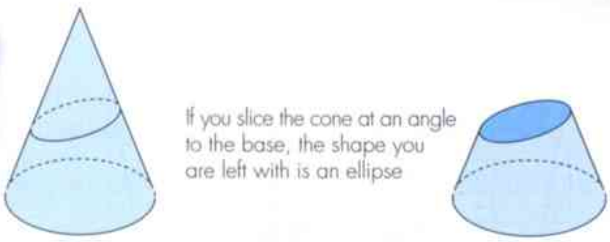
Why this chapter matters

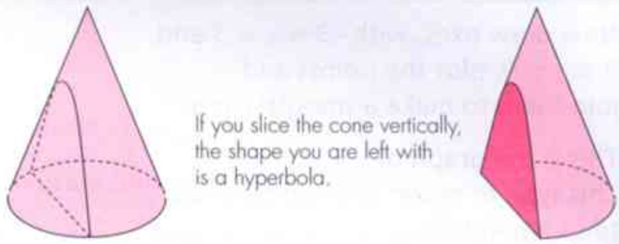
There are many curves that can be seen in everyday life. Did you know that all these curves can be represented mathematically?

Below are a few examples of simple curves that you may have noticed. Can you think of others?

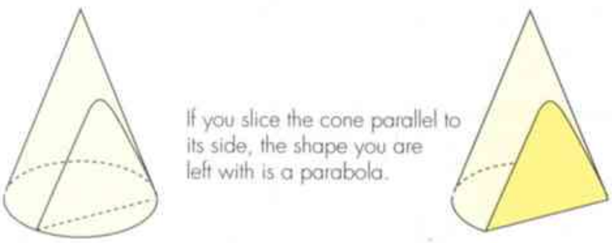
In mathematics, curves can take many shapes. These can be demonstrated using a cone, as shown on the right and below. If you make a cone out of modelling clay, you can see this for yourself. As you look at these curves, try to think of where you have seen them in your own life.



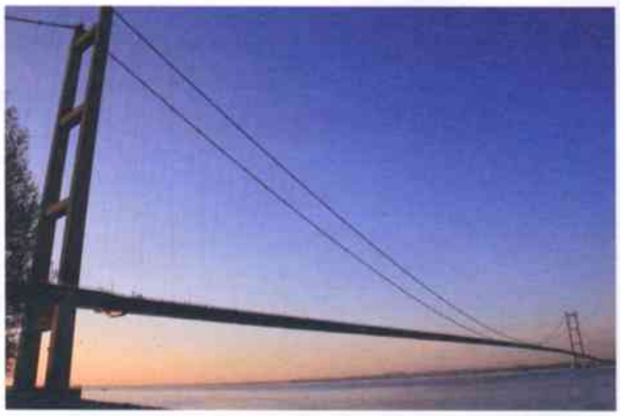




The curve that will be particularly important in this chapter is the parabola. Car headlights are shaped like parabolas.



All parabolas can be represented by **quadratic graphs**. In this chapter you will look at how to use **quadratic equations** to draw graphs that have this kind of curve.



The suspension cables on this bridge are also parabolas.

16.1 Quadratic graphs

A graph with a ∪ or ∩ shape is a quadratic graph.

A **quadratic graph** has an equation that involves a 'squared' term, such as x^2.

All of the following are **quadratic equations** and each would produce a quadratic graph.

$y = x^2$

$y = x^2 + 5$

$y = x^2 - 3x$

$y = x^2 + 5x + 6$

$y = x^2 + 2x - 5$

Example 1

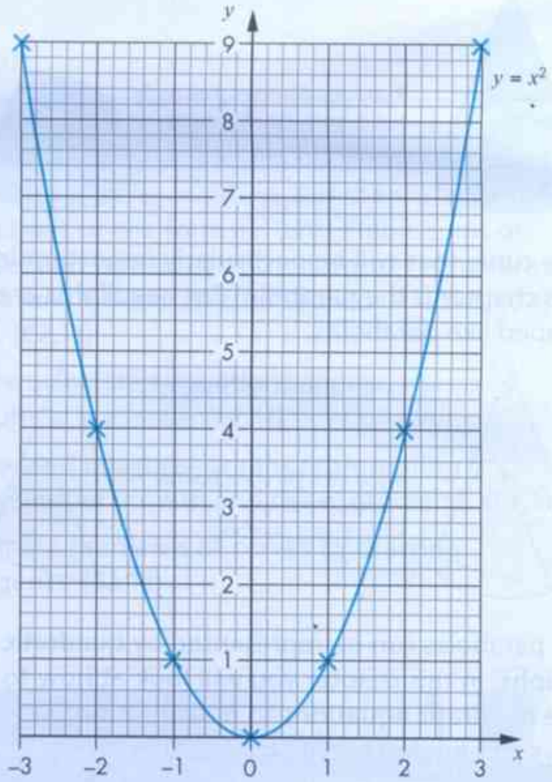
Draw the graph of $y = x^2$ for $-3 \leqslant x \leqslant 3$.

First make a table, as shown below.

x	−3	−2	−1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9

Now draw axes, with $-3 \leqslant x \leqslant 3$ and $0 \leqslant y \leqslant 9$, plot the points and join them to make a smooth curve.

This is the graph of $y = x^2$.
This type of graph is often referred to as a **parabola**.



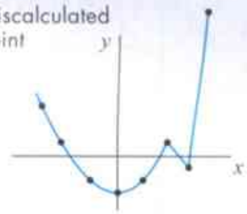
Here are some of the more common mistakes that you should try to avoid, when you are drawing a curve.

- When the points are too far apart, a curve tends to 'wobble'.

- Drawing curves in small sections leads to 'feathering'.

- The place where a curve should turn smoothly is drawn 'flat'.

- A curve is drawn through a point that, clearly, has been incorrectly plotted.



A quadratic curve drawn correctly will *always* be a smooth curve.

Here are some tips that will make it easier for you to draw smooth, curved graphs.

- If you are *right-handed*, turn your piece of paper or your exercise book round so that you draw from left to right. Your hand may be steadier this way than if you try to draw from right to left or away from your body.

  If you are *left-handed*, you may find drawing from right to left the more accurate way.
- Move your pencil over the points as a practice run without drawing the curve.
- Do one continuous curve and only stop at a plotted point.
- Use a *sharp* pencil and do not press too heavily, so that you may easily rub out mistakes.

Example 2

a Draw the graph of $y = x^2 + 2x - 3$ for $-4 \leqslant x \leqslant 2$.

b Use your graph to find the value of y when $x = 1.6$.

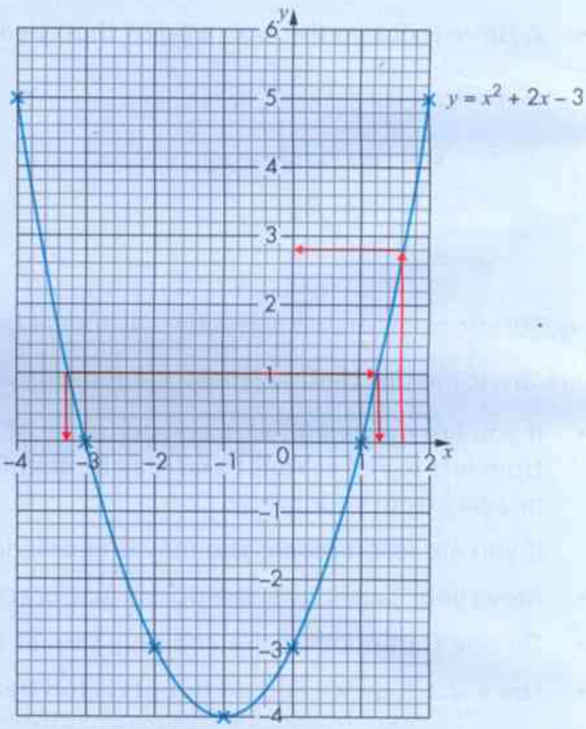
c Use your graph to find solve the equation $x^2 + 2x - 3 = 1$.

a Draw a table like this, to help work out each step of the calculation.

x	−4	−3	−2	−1	0	1	2
x^2	16	9	4	1	0	1	4
$+2x$	−8	−6	−4	−2	0	2	4
−3	−3	−3	−3	−3	−3	−3	−3
$y = x^2 + 2x - 3$	5	0	−3	−4	−3	0	5

Generally, you do not need to work out all values in a table. If you use a calculator, you need only to work out the y-value. The other rows in the table are just working lines to break down the calculation.

b To find the corresponding y-value for any value of x, you start on the x-axis at that x-value, go up to the curve, across to the y-axis and read off the y-value. This procedure is marked on the graph with arrows.



So when $x = 1.6$, $y = 2.8$.

c If $x^2 + 2x - 3 = 1$, this means that $y = 1$. This time, start at 1 on the y-axis and read off the two x-values that correspond to a y-value of 1.

Again, this procedure is marked on the graph with arrows.

So when $y = 1$, $x = -3.2$ or $x = 1.2$.

There are two possible values of x that make $x^2 + 2x - 3 = 1$.

Notice that the graph is symmetrical. The equation of the line of symmetry is $x = -1$, and the lowest point is (−1, −4).

The solutions of an equation like $x^2 + 2x - 3 = 0$ are called the **roots** of the equation.

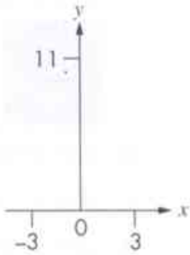
Look at the graph in Example 2. You can see that the roots of the equation $x^2 + 2x - 3 = 0$ are −3 and 1.

EXERCISE 16A

1 a Copy and complete the table for $y = x^2 + 2$.

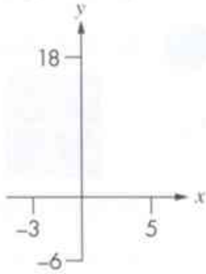
x	−3	−2	−1	0	1	2	3
$y = x^2 + 2$	11			2		6	

b Draw a graph of $y = x^2 + 2$ for $-3 \leqslant x \leqslant 3$.



2 a Copy and complete the table for $y = x^2 - 3x$ for $-3 \leqslant x \leqslant 5$.
Use your table to plot the graph.

x	−3	−2	−1	0	1	2	3	4	5
x^2						4			
$-3x$						−6			
y						−2			



b Use your graph to find the value of y when $x = 3.5$.

c What are the coordinates of the lowest point on the graph?

d What is the equation of the line of symmetry?

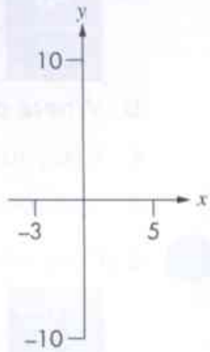
e Use your graph to solve the equation $x^2 - 3x = 5$.

Advice and Tips

You may find you do not need the second and third rows.

3 a Copy and complete the table for the graph of $y = x^2 - 2x - 8$ for $-3 \leqslant x \leqslant 5$. Use your table to plot the graph.

x	−3	−2	−1	0	1	2	3	4	5
y						−8			



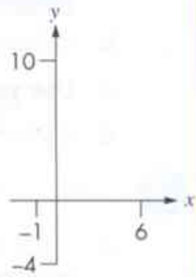
b Find the roots of the equation $x^2 - 2x - 8 = 0$

c Use your graph to find the value of y when $x = 0.5$.

d Use your graph to solve the equation $x^2 - 2x - 8 = 3$.

4 a Copy and complete the table for $y = x^2 - 5x + 4$ for $-1 \leqslant x \leqslant 6$.
Use your table to plot the graph.

x	−1	0	1	2	3	4	5	6
y		4			−2			



b What are the coordinates of the lowest point on the graph?

c What is the equation of the line of symmetry?

d Use your graph to find the value of y when $x = -0.5$.

e Find the roots of the equation $x^2 - 5x + 4 = 0$

f Use your graph to solve the equation $x^2 - 5x + 4 = 3$.

CORE

5 a Copy and complete the table for $y = x^2 + 2x - 1$ for $-4 \leqslant x \leqslant 2$.
Use your table to plot the graph.

x	−4	−3	−2	−1	0	1	2
y	7						

b Use your graph to find the approximate values of the roots of the equation $x^2 - 6x + 3 = 0$

c Use your graph to find the approximate values of the roots of the equation $2x - 1 = 0$

d Use your graph to find the approximate values of the roots of the equation $x^2 + 2x = 2$

6 a Copy and complete the table to draw the graph of $y = 12 - x^2$ for $-4 \leqslant x \leqslant 4$.

x	−4	−3	−2	−1	0	1	2	3	4
y				11				3	

b Use your graph to find the y-value when $x = 1.5$.

c Use your graph to solve the equation $12 - x^2 = 0$.

d Use your graph to solve the equation $12 - x^2 = 7$.

7 a Copy and complete the table to draw the graph of $y = x^2 + 4x$ for $-5 \leqslant x \leqslant 2$.

x	−5	−4	−3	−2	−1	0	1	2
x^2	25			4			1	
$+4x$	−20			−8			4	
y	5			−4			5	

b Where does the graph cross the x-axis?

c Use your graph to find the y-value when $x = -2.5$.

d Use your graph to solve the equation $x^2 + 4x = 3$.

8 a Copy and complete the table to draw the graph of $y = x^2 - 6x + 3$ for $-1 \leqslant x \leqslant 7$.

x	−1	0	1	2	3	4	5	6	7
y	10			−5			−2		

b Where does the graph cross the x-axis?

c Use your graph to find the y-value when $x = 3.5$.

d Use your graph to solve the equation $x^2 - 6x + 3 = 5$.

9 $y = 5x - x^2$

a Copy and complete this table of values.

x	−1	0	1	2	3	4	5	6
y								

b Draw a graph of $y = 5x - x^2$ for $-1 \leqslant x \leqslant 6$.

c What is the highest point on the graph?

d What is the equation of the line of symmetry on the graph?

e Solve the equation $5x - x^2 = 2$.

16.2 Turning points on a quadratic graph

E

Here is the equation of the quadratic graph from Example 2: $y = x^2 + 2x - 3$

In Section 13.5 you saw how to write expressions like $x^2 + 2x - 3$ in completed square form.

$y = x^2 + 2x - 3 = (x + 1)^2 - 1 - 3$

$= (x + 1)^2 - 4$

Now $(x + 1)^2 \geqslant 0$ and only equals 0 when $x = -1$

This means that $x^2 + 2x - 3$ has a minimum value of −4 when $x = -4$

This means that (−1, −4) on the graph of $y = x^2 + 2x - 3$

It is called a **turning point**.

It is a point where the graph changes direction.

Advice and Tips

Look at Section 13.5 if you have forgotten how to do this.

Example 3

a Find the turning point of the graph of $y = x^2 - 3x - 3$

b Sketch the graph of $y = x^2 - 3x - 3$

a Write $x^2 - 3x - 3$ in completed square form.

$x^2 - 3x - 3 = (x - 1.5)^2 - 1.5^2 - 3$

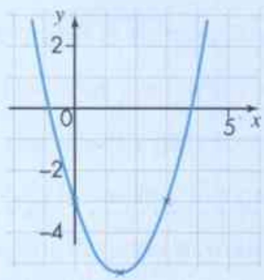
$= (x - 1.5)^2 - 5.25$

$(x - 1.5)^2 = 0$ when $x = 1.5$ and so the turning point is (1.5, −5.25)

b To find a point on the curve, let $x = 0$ and then $y = -3$

Hence (0, −3) is on the curve.

You can draw a sketch using (1.5, −5.25) and (0, −3)



Notice that, by symmetry, (3, −3) is also on the curve.

EXERCISE 16B

EXTENDED

1 The equation of a graph is $y = x^2 - 2x - 8$

- **a** Write $x^2 - 2x - 8$ in completed square form.
- **b** Find the coordinates of the turning point of the graph.
- **c** Solve the equation $x^2 - 2x - 8 = 0$
- **d** Hence sketch the graph of $y = x^2 - 2x - 8$

2 The equation of a graph is $y = x^2 + 10x + 21$

- **a** Write $x^2 + 10x + 21$ in completed square form.
- **b** Find the coordinates of the turning point of the graph.
- **c** Find the roots of the equation $x^2 + 10x + 21 = 0$
- **d** Sketch the graph of $y = x^2 + 10x + 21$

3
- **a** Write $x^2 - 7x + 10$ in completed square form.
- **b** Find the coordinates of the turning point of the graph of $y = x^2 - 7x + 10$
- **c** Solve the equation $x^2 - 7x + 10 = 0$
- **d** Sketch the graph of $y = x^2 - 7x + 10$

4 The equation of a graph is $y = x^2 - 6x + 12$

- **a** Find the intercept on the y-axis.
- **b** Find the turning point of the graph.
- **c** Sketch the graph of $y = x^2 - 6x + 12$
- **d** Explain why the graph shows that the equation $x^2 - 6x + 12 = 0$ has no solution.

5
- **a** Find the turning point of the graph of $y = x^2 + 20x + 40$
- **b** Sketch the graph of $y = x^2 + 20x + 40$

6 Find the smallest possible value of $x^2 - 25x + 100$

7 The equation of a curve is $y = x^2 + bx + c$ where b and c are constants.

The intercept on the y-axis is (0, 14).

The turning point of the graph is (5, –11)

Find the values of b and c.

16.3 Reciprocal graphs

A **reciprocal** equation has the form $y = \frac{a}{x}$.

Examples of reciprocal equations are: $y = \frac{1}{x}$ $y = \frac{4}{x}$ $y = -\frac{3}{x}$

All reciprocal graphs have a similar shape and some symmetry properties.

Example 4

Complete the table to draw the graph of $y = \frac{1}{x}$ for $-4 \leqslant x \leqslant 4$.

x	−4	−3	−2	−1	0	1	2	3	4
y									

Values are rounded to two decimal places, as it is unlikely that you could plot a value more accurately than this. The completed table looks like this.

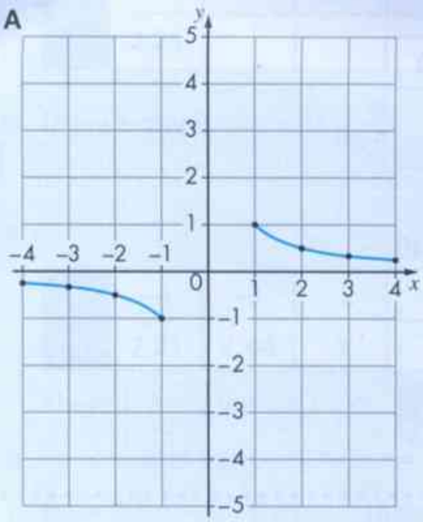
x	−4	−3	−2	−1	0	1	2	3	4
y	−0.25	−0.33	−0.5	−1	–	1	0.5	0.33	0.25

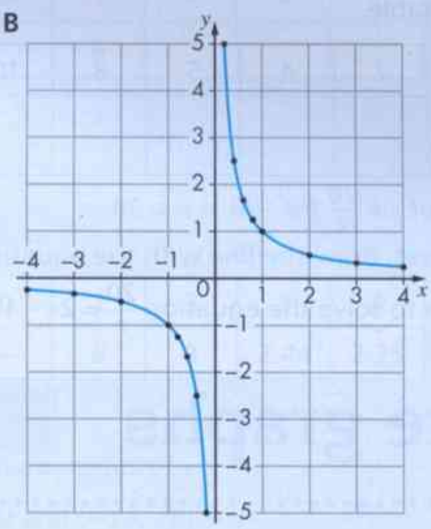
There is no value for $x = 0$ because 1/0 is undefined.

The graph plotted from these values is shown in **A**. This does not include much of the graph and does not show the properties of the reciprocal function. If you take x-values from −0.8 to 0.8 in steps of 0.2, you get the next table.

x	−0.8	−0.6	−0.4	−0.2	0.2	0.4	0.6	0.8
y	−1.25	−1.67	−2.5	−5	5	2.5	1.67	1.25

Plotting these points as well gives the graph in **B**.



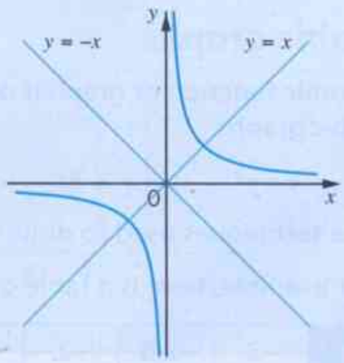


The graph in **B** shows these properties.

- The lines $y = x$ and $y = -x$ are lines of symmetry.
- The closer x gets to zero, the nearer the graph gets to the y-axis.
- As x increases, the graph gets closer to the x-axis.

The graph never actually touches the axes, it just gets closer and closer to them.

These properties are true for *all reciprocal graphs*.



EXERCISE 16C

CORE

1 **a** Copy and complete the table to draw the graph of $y = \frac{2}{x}$ for $-4 \leqslant x \leqslant 4$.

x	0.2	0.4	0.5	0.8	1	1.5	2	3	4
y	10		4				1		0.5

b Use your graph to find the y-value when $x = 2.5$.

c Use your graph to solve the equation $\frac{2}{x} = 7$.

d Use your graph to solve the equation $\frac{2}{x} = -1.25$.

2 **a** Copy and complete the table to draw the graph of $y = \frac{5}{x}$ for $-20 \leqslant x \leqslant 20$.

x	0.2	0.4	0.5	1	2	5	10	15	20
y	25		10		2.5				0.25

b On the same axes, draw the line $y = x + 10$.

c Use your graph to solve the equation $\frac{5}{x} = x + 10$.

3 Draw a graph of $y = \frac{-2}{x}$ for $-4 \leqslant x \leqslant 4$.

Use a table of values like the one in question **1**.

4 **a** Complete this table.

x	1	2	4	5	8	10	20
$\frac{20}{x}$							

b Draw a graph of $y = \frac{20}{x}$ for $-20 \leqslant x \leqslant 20$.

c On the same axes, draw the line with the equation $2x - 10$.

d Use your graph to solve the equation $\frac{20}{x} = 2x - 10$.

16.4 More graphs

E

Cubic graphs

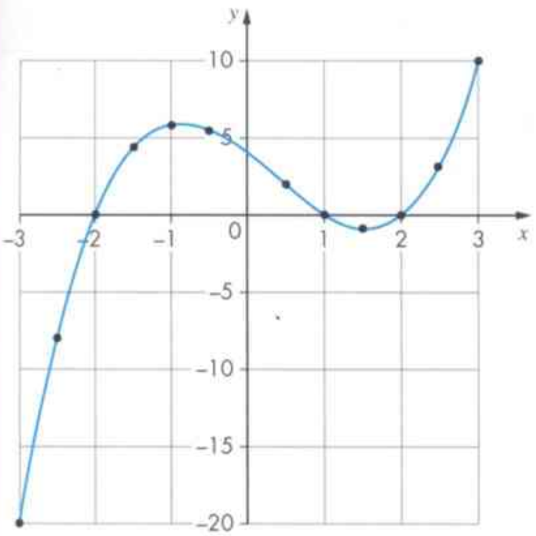
A **cubic** function or graph is one that contains a term in x^3. The following are examples of cubic graphs.

$y = x^3$ $\quad$ $y = x^3 + 3x$ $\quad$ $y = x^3 + x^2 + x + 1$

The techniques used to draw them are exactly the same as those for quadratic and reciprocal graphs.

For example, here is a table of values and graph of $y = x^3 - x^2 - 4x + 4$

x	−3	−2.5	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2	2.5	3
y	−20.00	−7.88	0.00	4.38	6.00	5.63	4.00	1.88	0.00	−0.88	0.00	3.38	10.00



This graph has two turning points.

These have approximate coordinates of (–1, 6) and (1.5, –0.9).

Example 5

a Complete a table of values for $y = 2 + \frac{4}{x^2}$

x	–4	–3	–2	–1	0	1	2	3	4
y	2.25					6			

b Draw a graph of $y = 2 + \frac{4}{x^2}$

a

x	–4	–3	–2	–1	0	1	2	3	4
y	2.25	2.44	3	6	–	6	3	2.44	2.25

There is no value for $x = 0$

b To see what happens near $x = 0$ find more values of y.

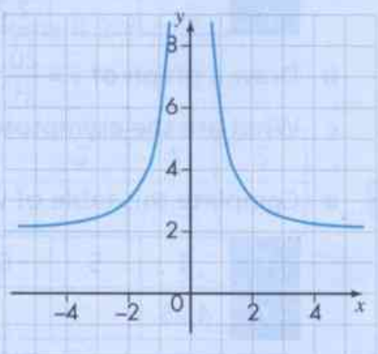
If $y = 0.5$ or -0.5 then $x = 18$ so (0.5, 18) and (–0.5, 18) are on the curve.

If x is a large positive or negative value, the curve is close to the line $x = 2$

The line $x = 2$ is called an asymptote to the curve.

If y is large, the curve is close to the y-axis (the line $x = 0$).

The y-axis is also an asymptote.



EXERCISE 16D

EXTENDED

1 Draw the graph of $y = x^3$ for $-2 \leqslant x \leqslant 2$.

2 **a** Complete the table to draw the graph of $y = 0.5x^3$ for $-2.5 \leqslant x \leqslant 2.5$.

x	−2.5	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2	2.5
y			−1.69			0.00					7.81

b Use your graph to solve the equation $0.5x^3 = 6$.

3 **a** Complete the table to draw the graph of $y = x^3 + 3$ for $-2.5 \leqslant x \leqslant 2.5$.

x	−2.5	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2	2.5
y	−12.63			2.00		3.00	3.13			11.00	

b Use your graph to solve the equation $x^3 + 3 = 0$.

4 **a** Complete the table to draw the graph of $y = x^3 - 2x + 5$ for $-2 \leqslant x \leqslant 2$.

x	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
y	1.00	4.63			5.00	4.13			

b Use your graph to solve the equation $x^3 - 2x + 5 = 3$.

c Use your graph to find the root of the equation $x^3 - 2x - 2 = 0$

d Use the graph to find the approximate coordinates of the turning points.

5 **a** Complete this table of values for $y = x^3 - x^2 - 6x$

x	−3	−2	−1	0	1	2	3	4
y		0			−6			

b Draw a graph of $y = x^3 - x^2 - 6x$

c Use your graph to solve the equation $x^3 - x^2 - 6x + 5 = 0$

d Use your graph to find the coordinates of the turning points on the graph

6 **a** Complete this table of values for $y = \frac{20}{x^2}$. Give the values to 2 decimal places.

x	1	2	3	4	5
y			2.22		

b Draw a graph of $y = \frac{20}{x^2}$ for $-5 \leqslant x \leqslant 5$.

c What are the asymptotes of the curve?

7 **a** Complete this table of values for $y = \frac{100}{x^2} - 0.5x$.

x	4	5	6	7	8
y	4.25			−1.46	

b Draw a graph of $y = \frac{100}{x^2} - 0.5x$ for $4 \leqslant x \leqslant 8$.

c Use the graph to solve the equation $\frac{100}{x^2} - 0.5x = 0$.

8 This is a sphere.

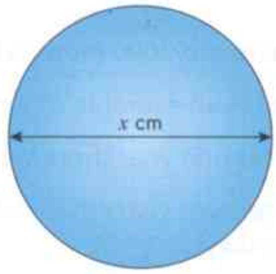
The diameter is x cm.

The volume is $\frac{1}{6}\pi x^3$ cm³.

a Draw a graph of $y = \frac{1}{6}\pi x^3$ for $6 \leqslant x \leqslant 10$.

Use a scale of 2 cm to 1 unit on the x-axis and 2 cm to 100 units on the y-axis.

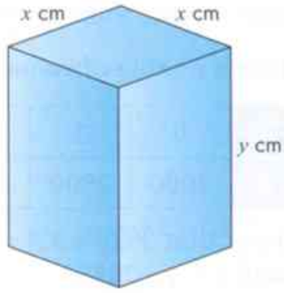
b Use your graph to find the diameter of a sphere with a volume of 300 cm³.



9 The volume of this cuboid is 1000 cm³.

The top is a square of side x cm.

The height is y cm.



a Show that $y = \frac{1000}{x^2}$.

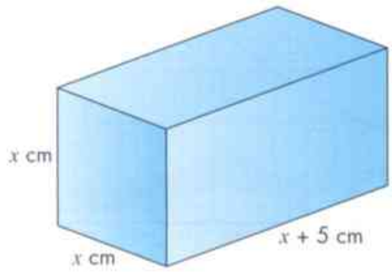
b Draw a graph of $y = \frac{1000}{x^2}$ for $5 \leqslant x \leqslant 10$.

Use a scale of 2 cm to 1 unit on the x-axis and 2 cm to 10 units on the y-axis

10 This is a cuboid. The end is a square of side x cm.

The length is 5 cm greater than the side of the square end.

The volume of the cuboid is $x^2(x + 5)$ cm³.



a Draw a graph of $y = x^2(x + 5)$ for $0 \leqslant x \leqslant 6$.

Use a scale of 2 cm to 1 unit on the x-axis and 2 cm to 100 units on the y-axis

b Use your graph to find the sides of the cuboid when the volume is 200 cm³.

11 **a** Complete this table of values of y = $x + 4/x$

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
y		−5					5				5.8

b Draw the graph of $y = x + 4/x$

c Draw the line $y = x$ on your graph.

d Explain why $y = x$ is an asymptote to the curve.

e Find another asymptote.

16.5 Exponential graphs

There are 2000 monkeys in a forest. Scientists say that the numbers are increasing by 40% each year.

This is an example of **exponential growth**. The multiplier is 1.4.

After one year there will be 2000 × 1.4 = 2800 monkeys.

After two years there will be 2800 × 1.4 or $2000 \times 1.4^2 = 3920$ monkeys.

After x years there will be 2000×1.4^x monkeys.

Advice and Tips

Ignore the decimal fraction, you need only consider whole monkeys.

Here is a table of values.

x	0	1	2	3	4
y	2000	2800	3920	5488	7683

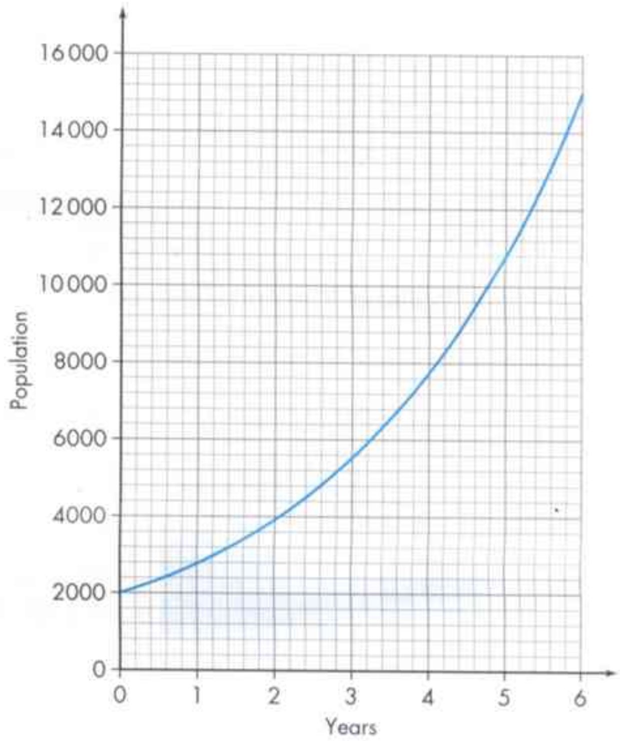
Check that 3920 × 1.4 = 5488 and 5488 × 1.4 = 7683.

A graph shows how the population of monkeys changes over time.

The graph shows that the population reaches 10 000 after nearly five years.

Graphs of exponential growth always have this shape.

As the value of x increases, the value of y increases and the graph gets steeper.



Example 6

Atmospheric pressure is measure in units called hectopascals (hPa).

It decreases as height above sea level increases.

At sea level atmospheric pressure is 1000 hPa.

It decreases by 12% for every kilometre increase in height.

a Draw a graph to show how atmospheric pressure changes, up to a height of six kilometres.

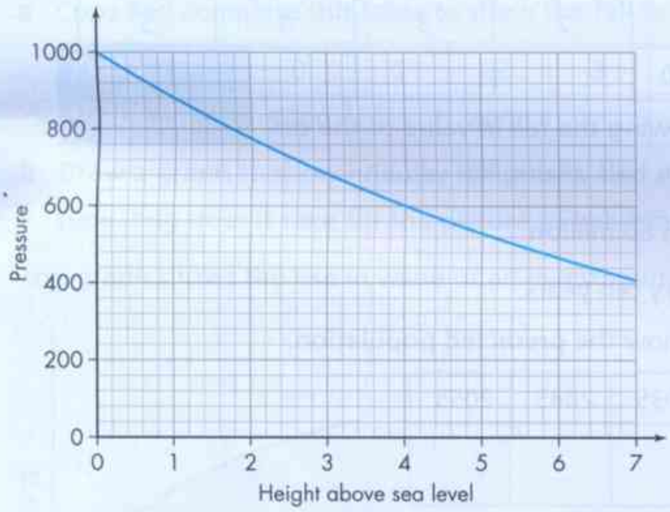
b At what height is the atmospheric pressure half the value at sea level?

a The multiplier for a decrease of 12% is 0.88. 100% − 12% = 88% = 0.88

At a height of x km the pressure is 1000×0.8^x hPa.

Height above sea level x (km)	Pressure y (hPa)
0	1000
1	$1000 \times 0.88 = 880$
2	$1000 \times 0.88^2 = 774$
3	$1000 \times 0.88^3 = 681$
4	$681 \times 0.88 = 600$
5	$600 \times 0.88 = 528$
6	464

This graph shows the values.



b Half the pressure at sea level is 500 mPa.

From the graph, when $y = 500$, $x = 5.4$.

The height is 5.4 km.

The graph in the example shows **exponential decay**.

As the value of x increases, the value of y decreases and the graph gets less steep.

EXTENDED

EXERCISE 16E

1 A company buys some new machinery for $96 000.

The value of the machinery halves every year.

a Copy and complete this table.

Years	0	1	2	3	4
Value (thousand dollars)	96				

b Draw a graph to show how the value falls.

2 The price of a car is $25 000 when it is new.

The value falls by 20% per year.

a Show that the value after one year is $20 000.

b Copy and complete this table.

Years	0	1	2	3	4	5
Value ($)	25 000	20 000				

c Use the table to draw a graph showing the fall in value of the car.

d How long is it until the car is worth half its original value?

3 The population of a country in 2015 is 60 million.

The population increases by 15% every ten years.

a Copy and complete this table to show the predicted population.

Years	2015	2025	2035	2045	2055
Population (millions)	60	69			

b Draw a graph to show how the population changes.

c Estimate the year when the population will reach 100 million.

4 Mira has shares worth $500.

Their value increases by 30% each year.

a Copy and complete this table.

Years	0	1	2	3	4	5
Value ($)	500					1856

b Draw a graph to show how the value of the shares increases.

c How long does it take for the value of the shares to double?

5 There are 20 mice in a population.

Mice breed very quickly. The number of mice doubles every month.

a Draw a graph to show how the number of mice increases over six months.

b How long will it be until there are 200 mice?

6 Abram puts $400 in a bank.

After one year it is worth $480 dollars.

a What is the annual percentage increase?

b The value increases exponentially. Draw a graph to show how the value increases over four years.

c Use the graph to find the value after 2.5 years.

7 Marcus buys a picture for $5000.

The value is decreasing exponentially by 10% a year.

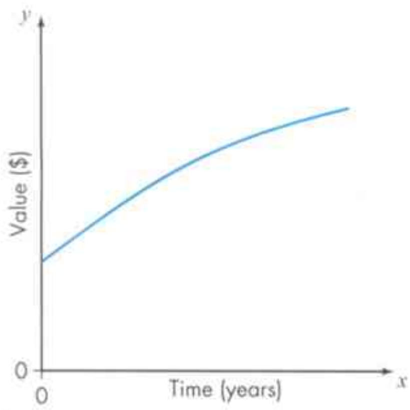
a Copy and complete this table to show the fall in value of the picture.

Years	0	1	2	3	4	5
Value ($)	5000		4050			2952

b Draw a graph to show how the value falls over eight years.

c How long does it take for the picture to halve in value?

8 This graph shows the rise in value of an investment.

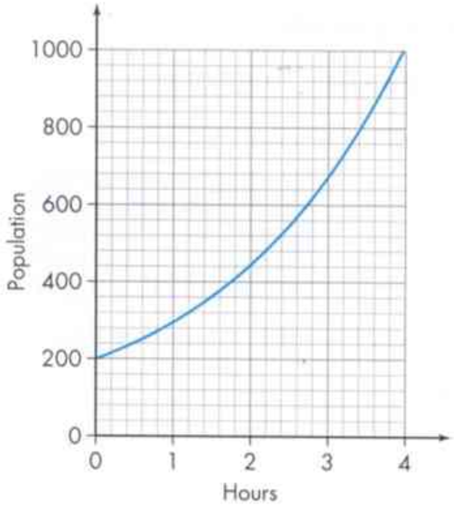


Mario says, 'The value increases all the time so this is exponential growth.'

Explain why Mario is not correct.

EXTENDED

9 The graph shows the number of bacteria in a population that is growing exponentially.



a What was the initial population?

b Work out the percentage growth of the population each hour.

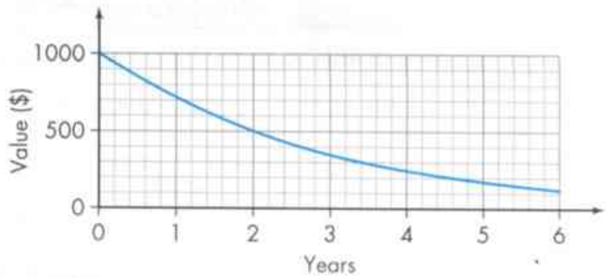
c How long did the population take to triple in size?

10 Barak buys a motor bike for $1000.

The graph shows the fall in value of the bike.

a Show that the value halves every two years.

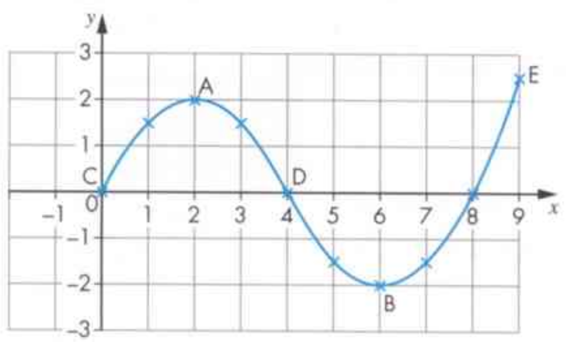
b Work out the percentage fall in the value of the bike every year.



16.6 Estimating gradients

E

The **gradient** of a curve varies from point to point. At points on this curve to the left of A or to the right of B the gradient is positive.



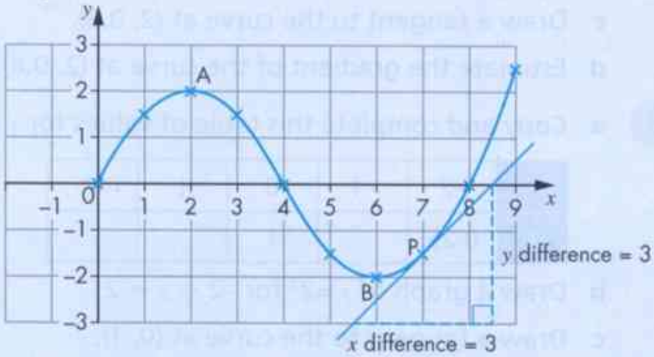
Between A and B the gradient is negative.

The curve is steepest near C, D or E and the gradient there will have the greatest magnitude.

You can estimate the gradient at any point on a curve by drawing a **tangent** at that point. This is a straight line that touches the curve at that point and has the same gradient. You can find the gradient of the tangent, which is a straight line, by drawing a triangle.

Example 7

Find the gradient at the point P with the coordinates (7, –1.5).



Draw a tangent at point P.

Draw a triangle and find the differences between the x-coordinates and y-coordinates.

$$\text{gradient} = \frac{\text{difference in } y}{\text{difference in } x} = \frac{3}{3} = 1$$

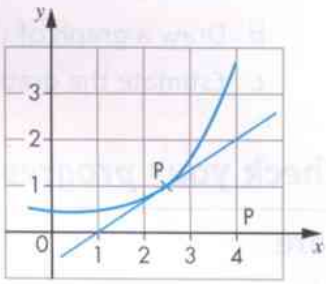
The gradient of the curve at P is 1.

Note that at A and B the gradient is 0.

EXERCISE 16F

1 The straight line is a tangent to the curve at the point P with coordinates (2.5, 1).

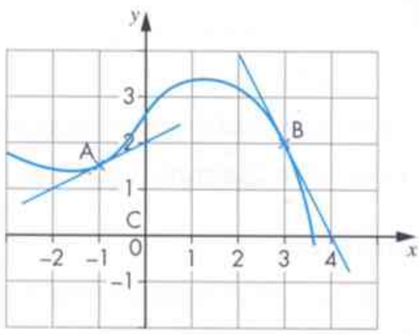
Find the gradient of the curve at P.



2 A and B have coordinates (–1, 1.5) and (3, 2).

Tangents to the curve have been drawn at A and B.

Calculate the gradient of the curve at A and at B.



3 a Use this table of values to draw a graph of y for $0 \leqslant x \leqslant 4$.

x	0	1	2	3	4
y	2	1.5	2	3.5	6

b Draw a tangent to the curve at the point (3, 3.5).

c Estimate the gradient of the curve at the point (3, 3.5).

d At which point is the gradient of the curve 0?

EXTENDED

EXTENDED

4 a Copy and complete this table of values for $0.1x^3$.

x	0	0.5	1	1.5	2	2.5	3
$0.1x^3$		0.01	0.1		0.8		

b Draw a graph of $y = 0.1x^3$ for $0 \leqslant x \leqslant 3$.

c Draw a tangent to the curve at (2, 0.8).

d Estimate the gradient of the curve at (2, 0.8).

5 a Copy and complete this table of values for $y = 2^x$.

x	−2	−1	0	1	2
y	0.25		1		

b Draw a graph of $y = 2^x$ for $-2 \leqslant x \leqslant 2$.

c Draw a tangent to the curve at (0, 1).

d Estimate the gradient of the curve at (0, 1).

6 a Copy and complete this table of values for $\frac{5}{x}$.

x	2	3	4	5	6
$\frac{5}{x}$			1.25		0.83

b Draw a graph of $y = \frac{5}{x}$ for $2 \leqslant x \leqslant 6$.

c Estimate the gradient of the curve at (4, 1.25).

Check your progress

Core

- I can construct tables of values of functions of the form:
  - $+x^2 + ax + b$
  - a/x
- I can draw and interpret graphs of such functions
- I can solve quadratic equations approximately using a graph

Extended

- I can construct tables and draw graphs of functions of the form ax^n and simple sums of these, where $n = -2, -1, 0, 1, 2$ or 3
- I can construct tables and draw graphs of functions of the form
- I can solve approximately equations related to such graphs
- I can draw and interpret graphs representing exponential growth or decay
- I can recognise, sketch and interpret linear, quadratic, cubic, reciprocal and exponential graphs
- I can recognise turning points and asymptotes on a graph
- I can estimate the gradient of a curve by drawing a tangent

Chapter 17

Number sequences

Topics	Level	Key words
1 Patterns in number sequences	CORE	sequence, term, difference, consecutive
2 The *n*th term of a sequence	CORE	coefficient, linear sequence, *n*th term, quadratic sequence, cubic sequence
3 General rules from patterns	CORE	rule, patterns
4 Further sequences	EXTENDED	exponential sequence

In this chapter you will learn how to:

CORE

Calculate:

- Continue a given number sequence. (C2.7 and E2.7)
- Recognise patterns in sequences and relationships between different sequences, including the term to term rule. (C2.7 and E2.7)
- Find the *n*th term for linear sequences, and for simple quadratic and cubic sequences. (C2.7 and E2.7)

EXTENDED

- Find the *n*th term of linear, quadratic, cubic and exponential sequences and simple combinations of these. (E2.7)

Why this chapter matters

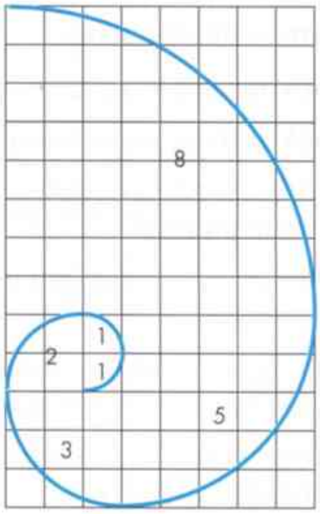
Patterns often appear in numbers. Prime numbers, square numbers and multiples all form patterns. Mathematical patterns also appear in nature.

There are many mathematical patterns that appear in nature. The most famous of these is probably the **Fibonacci** series.

1 1 2 3 5 8 13 21 ...

This is formed by adding the two previous terms to get the next term.

The sequence was discovered by the Italian, Leonardo Fibonacci, in 1202, when he was investigating the breeding patterns of rabbits!



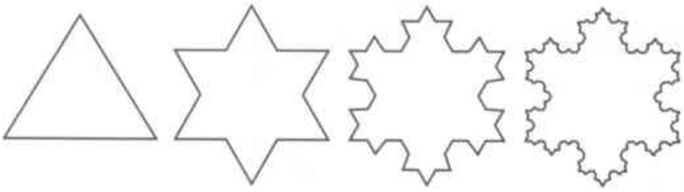
Since then, the pattern has been found in many other places in nature. The spirals found in a nautilus shell and in the seed heads of a sunflower plant also follow the Fibonacci series.

Fractals form another kind of pattern.

Fractals are geometric patterns that are continuously repeated on a smaller and smaller scale.

A good example of a fractal is this: start with an equilateral triangle and draw an equilateral triangle, a third the size of the original, on the middle of each side. Keep on repeating this and you will get an increasingly complex-looking shape.

The pattern shown here is called the Koch snowflake. It is named after the Swedish mathematician, Helge von Koch (1870–1924).



Fractals are commonly found in nature, a good example being the complex patterns found in plants, such as the leaves of a fern.

Chapter 17 . Topic 1

17.1 Patterns in number sequences

A number **sequence** is an ordered set of numbers with a rule for finding every number in the sequence. The rule that takes you from one number to the next could be a simple addition or multiplication, but often it is more tricky than that. You always need to look very carefully at the pattern of a sequence.

Each number in a sequence is called a **term** and is in a certain position in the sequence.

Look at these sequences and their rules.

3, 6, 12, 24, … doubling the previous term each time … 48, 96, … the term to term rule is 'multiply by 2'

2, 5, 8, 11, … adding 3 to the previous term each time … 14, 17, … the term to term rule is 'add 3'

1, 10, 100, 1000, … multiplying the previous term by 10 each time … 10000, 100000 the term to term rule is 'multiply by 10'

80, 73, 66, 59, … subtracting 7 from the previous term each time … 52, 45, … the term to term rule is 'subtract 7'

These are all quite straightforward once you have looked for the link from one term to the next (**consecutive** terms). This link is called the term to term rule.

Differences

For some sequences you need to look at the **differences** between consecutive terms to determine the pattern.

Example 1

Find the next two terms of the sequence 1, 3, 6, 10, 15, … .

Looking at the differences between consecutive terms:

1	3	6	10	15
↑ 2	↑ 3	↑ 4	↑ 5	

So the sequence continues as follows.

1	3	6	10	15	21	28
↑ 2	↑ 3	↑ 4	↑ 5	+6	+7	

So the next two terms are 21 and 28.

The differences usually form a number sequence of their own, so you need to find the *sequence of the differences* before you can expand the original sequence.

EXERCISE 17A

1 Look at these number sequences. Write down the next three terms in each and state the term to term rule.

a 1, 3, 5, 7, …
b 2, 4, 6, 8, …
c 5, 10, 20, 40, …
d 1, 3, 9, 27, …
e 4, 10, 16, 22, …
f 3, 8, 13, 18, …
g 2, 20, 200, 2000, …
h 7, 10, 13, 16, …
i 150, 141, 132, 123, …
j 5, 15, 45, 135, …
k 400, 200, 100, 50, …
l 1, 5, 25, 125, …

2 By considering the differences in these sequences, write down the next two terms in each one.

a 1, 2, 4, 7, 11, …
b 1, 2, 5, 10, 17, …
c 1, 3, 7, 13, 21, …
d 1, 4, 10, 19, 31, …
e 1, 9, 25, 49, 81, …
f 1, 2, 7, 32, 157, …
g 1, 3, 23, 223, 2223, …
h 1, 2, 4, 5, 7, 8, 10, …
i 2, 3, 5, 9, 17, …
j 3, 8, 18, 33, 53, …

3 Look at the sequences below. Find the term to term rule for each sequence and write down its next three terms.

a 3, 6, 12, 24, …
b 3, 9, 15, 21, 27, …
c 128, 64, 32, 16, 8, …
d 50, 47, 44, 41, …
e 2, 5, 10, 17, 26, …
f 5, 6, 8, 11, 15, 20, …
g 5, 7, 8, 10, 11, 13, …
h 40, 37, 34, 31, 28, …
i 1, 3, 6, 10, 15, 21, …
j 1, 2, 3, 4, …
k 100, 20, 4, 0.8, …
l 1, 0.5, 0.25, 0.125, …

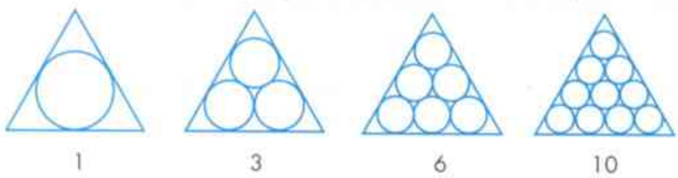
4 Look carefully at each number sequence. Find the next two numbers in the sequence and try to explain the pattern.

a 1, 1, 2, 3, 5, 8, 13, …
b 1, 4, 9, 16, 25, 36, …
c 3, 4, 7, 11, 18, 29, …
d 1, 8, 27, 64, 125, …

Advice and Tips

These patterns do not go up by the same value each time so you will need to find another connection between the terms.

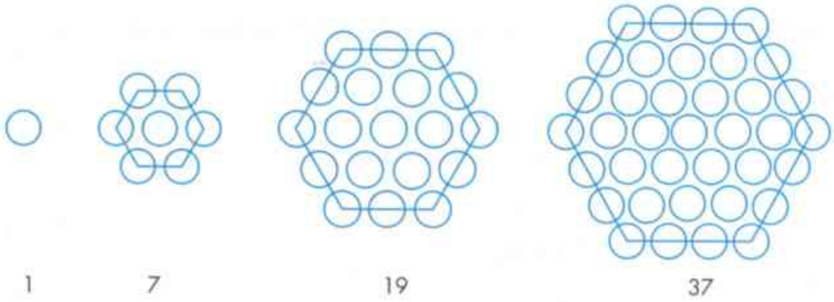
5 The sequence of triangular numbers builds up like this.



Find the next four triangular numbers.

CORE

6 The sequence of hexagonal numbers builds up like this.



Find the next three hexagonal numbers.

7 The first term that these two sequences have in common is 17.

8, 11, 14, 17, 20, ...

1, 5, 9, 13, 17, ...

What are the next two terms that the two sequences have in common?

8 Here are the first few terms of two sequences.

2, 5, 8, 11, 14, ...

3, 6, 9, 12, 15, ...

Will the two sequences ever have a term in common? Yes or no?
Justify your answer.

9 Here are two sequences.

100, 95, 90, 85, ...

3, 10, 17, 24, ...

Write down all the numbers that will occur in **both** sequences.

17.2 The *n*th term of a sequence

Finding the rule

When using a number sequence, you sometimes need to know, say, its 50th term, or even a higher term in the sequence. To work out these terms, you need to find the rule that produces the sequence in its general form.

It may be helpful to look at the problem backwards. That is, take a rule and see how it produces a sequence. The rule is given for the general term, which is called the ***n*th term**.

Example 2

The nth term of a sequence is $3n + 1$, where $n = 1, 2, 3, 4, 5, 6, \ldots$.
Write down the first five terms of the sequence.

Substituting $n = 1, 2, 3, 4, 5$ in turn:

$(3 \times 1 + 1), (3 \times 2 + 1), (3 \times 3 + 1), (3 \times 4 + 1), (3 \times 5 + 1), \ldots$
4 7 10 13 16

So the sequence is 4, 7, 10, 13, 16, ...

Notice that in Example 2 the difference between each term and the next is always 3, which is the **coefficient** of n (the number attached to n). Also, the constant term is the difference between the first term and the coefficient, that is, $4 - 3 = 1$.

EXERCISE 17B

CORE

1 Here are the nth terms of some sequences. Write down the first five terms of each sequence.

a $2n + 1$ for $n = 1, 2, 3, 4, 5$
b $3n - 2$ for $n = 1, 2, 3, 4, 5$
c $5n + 2$ for $n = 1, 2, 3, 4, 5$
d n^2 for $n = 1, 2, 3, 4, 5$
e $n^2 + 3$ for $n = 1, 2, 3, 4, 5$
f $20 - 2n$ for $n = 1, 2, 3, 4, 5$

2 Write down the first five terms of the sequence that has as its nth term:

a $n + 3$
b $3n - 1$
c $5n - 2$
d $n^2 - 1$
e $4n + 5$
f $45 - 3n$

Advice and Tips

Substitute numbers into the expressions until you can see how the sequence works.

3 The nth term of a sequence is $100 - 6n$.

a Work out the first four terms.

b Work out the first term that is less than zero.

4 Write down the first four terms of the sequence for which the nth term is:

a n^2
b $n^2 + 2$
c $2n^2$
d $4n^2 - 1$
e $200 - n^2$
f $\frac{n^2}{4}$

5 Write down the first four terms of a sequence for which the nth term is:

a n^3
b $n^3 + 1$
c $n^3 - 2$
d $3n^3 - 1$
e $\frac{n^3}{2}$
f $109 - n^3$

6 A haulage company uses this formula to calculate the cost of transporting n pallets.

For $n \leqslant 5$, the cost will be \$$(40n + 50)$
For $6 \leqslant n \leqslant 10$, the cost will be \$$(40n + 25)$
For $n \geqslant 11$, the cost will be \$$40n$

a How much will the company charge to transport 7 pallets?

b How much will the company charge to transport 15 pallets?

CORE

c A company is charged \$170 for transporting pallets. How many pallets did they transport?

d Another haulage company uses the formula \$50n to calculate the cost for transporting n pallets.

At what value of n do the two companies charge the same amount?

7 The nth term of a sequence is $3n + 7$.
The nth term of another sequence is $4n - 2$.

These two series have several terms in common but only one term that is common and has the same position in the sequence.

Without writing out the sequences, show how you can tell, using the expressions for the nth term, that this is the 9th term.

Finding the nth term of a linear sequence

In a **linear sequence** the *difference* between one term and the next is always the same.

For example:

2, 5, 8, 11, 14, … difference of 3

The nth term of this sequence is given by $3n - 1$.

Here is another linear sequence.

5, 7, 9, 11, 13, … difference of 2

The nth term of this sequence is given by $2n + 3$.

So, you can see that the nth term of a linear sequence is *always* of the form $An + b$, where:

- A, the coefficient of n, is the difference between each term and the next term (**consecutive** term)
- b is the difference between the first term and A.

Example 3

From the sequence 5, 12, 19, 26, 33, … , find:

a the nth term

b the 50th term.

a The difference between consecutive terms is 7. So the first part of the nth term is $7n$.
Subtract the difference, 7, from the first term, 5, which gives $5 - 7 = -2$.
So the nth term is given by $7n - 2$.

b Now find the 50th term by substituting $n = 50$ into the rule, $7n - 2$.

50th term $= 7 \times 50 - 2 = 350 - 2$

$= 348$

Quadratic and cubic sequences

The square numbers are:

1, 4, 9, 16, 25, 36, …

The nth square number is n^2.

So, for example, the 15th square number is $15^2 = 225$.

The cube numbers are:

1, 8, 27, 64, 125, …

The nth cube number is n^3.

So, for example, the 7th cube number is $7^3 = 7 \times 7 \times 7 = 343$.

Other sequences can be formed from square and cube numbers.

Example 4

Find the nth term of this sequence.
5, 8, 13, 20, 29, …

The differences between terms are 3, 5, 7, 9, …

These are not the same, so it is *not* a linear sequence.

Compare the sequence with the square numbers.

5	8	13	20	29	…
1	4	9	16	25	…

You can see that each term is four more than a square number.

The nth term of this sequence is $n^2 + 4$.

A sequence that is based on square numbers is called a **quadratic sequence**.

Example 5

Find the nth term of this sequence.
0, 7, 26, 63, 124

The differences are not equal. It is not linear.

Compare the terms with the square numbers and the cube numbers.

Sequence	0	7	26	63	124	…
Square numbers	1	4	9	16	25	…
Cube numbers	1	8	27	64	125	…

Each term is one less than a cube number.

The nth term of the sequence is $n^3 - 1$.

A sequence that is based on cube numbers is called a **cubic sequence**.

EXERCISE 17C

CORE

1 Find the next two terms and the nth term in each of these linear sequences.

a 3, 5, 7, 9, 11, … **b** 5, 9, 13, 17, 21, …
c 8, 13, 18, 23, 28, … **d** 2, 8, 14, 20, 26, …
e 5, 8, 11, 14, 17, … **f** 2, 9, 16, 23, 30, …
g 27, 25, 23, 21, 19, … **h** 42, 38, 34, 30, 26, …
i 2, 5, 8, 11, 14, … **j** 2, 12, 22, 32, …
k 8, 12, 16, 20, … **l** 4, 9, 14, 19, 24, …

Advice and Tips

Remember to look at the differences and the first term.

2 Find the nth term and the 50th term in each of these linear sequences.

a 4, 7, 10, 13, 16, … **b** 7, 9, 11, 13, 15, …
c 3, 8, 13, 18, 23, … **d** 1, 5, 9, 13, 17, …
e 2, 10, 18, 26, … **f** 5, 6, 7, 8, 9, …
g 6, 11, 16, 21, 26, … **h** 3, 11, 19, 27, 35, …
i 1, 4, 7, 10, 13, … **j** 21, 24, 27, 30, 33, …
k 12, 19, 26, 33, 40, … **l** 1, 9, 17, 25, 33, …

3 For each sequence **a** to **j**, find:

i the nth term **ii** the 100th term.

a 5, 9, 13, 17, 21, … **b** 3, 5, 7, 9, 11, 13, …
c 4, 7, 10, 13, 16, … **d** 8, 10, 12, 14, 16, …
e 9, 13, 17, 21, … **f** 6, 11, 16, 21, …
g 0, 3, 6, 9, 12, … **h** 2, 8, 14, 20, 26, …
i 7, 15, 23, 31, … **j** 25, 27, 29, 31, …

4 An online CD retail company uses this price chart. The company charges a standard basic price for a single CD, including postage and packing.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Charge ($)	10	18	26	34	42	49	57	65	73	81	88	96	104	112	120

a Using the charges for 1 to 5 CDs, work out an expression for the nth term.
b Using the charges for 6 to 10 CDs, work out an expression for the nth term.
c Using the charges for 11 to 15 CDs, work out an expression for the nth term.
d What is the basic charge for a CD?

5 Here are some quadratic sequences. Work out the nth term of each one.

a 1, 4, 9, 16, 25, … **b** 3, 6, 11, 18, 27 …
c 2, 8, 18, 32, 50, … **d** 0, 3, 8, 15, 24

6 Here are some cubic sequences. Work out the nth term of each one.

a 1, 8, 27, 64, …
b 11, 18, 37, 74, …
c 0.5, 4, 13.5, 32, …
d 10, 80, 270, 640, …

7 Work out the nth term of each of these sequences.

a 6, 7, 8, 9, 10, …
b 6, 9, 14, 21, 30, …
c 6, 13, 32, 69, 130
d 6, 11, 16, 21, 26, …
e 5, 20, 45, 80, 125, …
f 5, 40, 135, 320, 625, …

17.3 General rules from patterns

Many problem-solving situations that you are likely to meet involve number sequences. So you need to be able to formulate general **rules** from given number **patterns**.

Example 6

The diagram shows a pattern of squares building up.

a How many squares will there be in the nth pattern?

b Which pattern has 99 squares in it?

a First, build up a table for the patterns.

Pattern number	1	2	3	4	5
Number of squares	1	3	5	7	9

Looking at the **difference** between consecutive patterns, you can see it is always two squares. So, use $2n$.

Subtract the difference 2 from the first number, which gives $1 - 2 = -1$.

So the number of squares on the base of the nth pattern is $2n - 1$.

b Now find n when $2n - 1 = 99$:

$2n - 1 = 99$

$2n = 99 + 1 = 100$

$n = 100 \div 2 = 50$

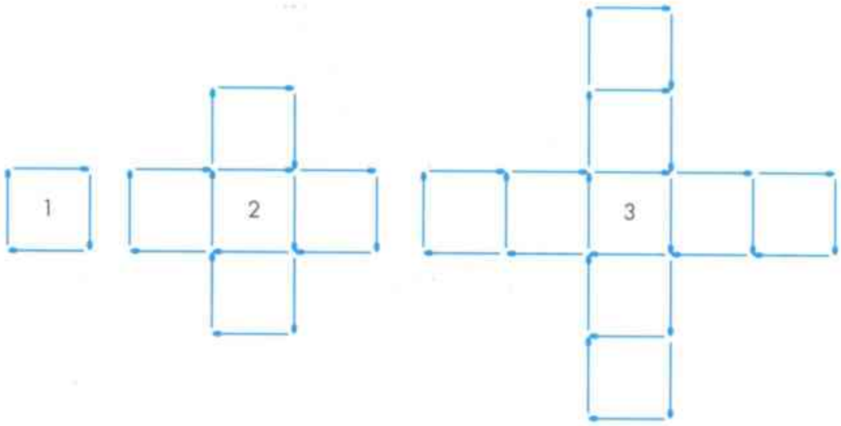
The pattern with 99 squares is the 50th.

When you are trying to find a general rule from a sequence of diagrams, always set up a table to connect the pattern number with the number of the variable (squares, matches, seats, etc.) for which you are trying to find the rule. Once you have set up the table, it is easy to find the nth term.

CORE

EXERCISE 17D

1 A pattern of squares is built up from matchsticks as shown.



a Draw the fourth diagram.

b Copy and complete this table.

Pattern number	1	2	3	4
Number of squares	1	5	9	

c How many squares are in the nth diagram?

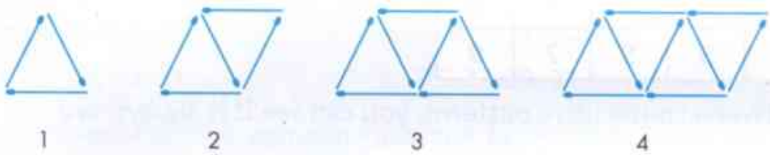
d How many squares are there in the 25th diagram?

e With 200 squares, which is the biggest diagram that could be made?

Advice and Tips

Write out the number sequences to help you see the patterns.

2 A pattern of triangles is built up from matchsticks.



a Draw the fifth set of triangles in this pattern.

b Copy and complete this table.

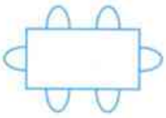
Pattern number	1	2	3	4	5
Number of matches	3				

c How many matchsticks are needed for the nth set of triangles?

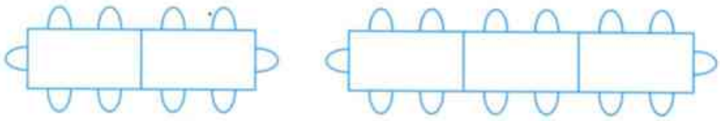
d How many matchsticks are needed to make the 60th set of triangles?

e If there are only 100 matchsticks, which is the largest set of triangles that could be made?

3 A conference centre had tables each of which could sit six people.



When put together, the tables could seat people as shown.



a How many people could be seated at four tables put together this way?

b Copy and complete this table.

Number of tables	1	2	3	4
Number of seats	6			

c How many people could be seated at n tables put together in this way?

d At a conference, there were 50 people who wished to use the tables in this way.

How many tables would they need?

4 Prepacked fencing units come in the shape shown, made of four pieces of wood.

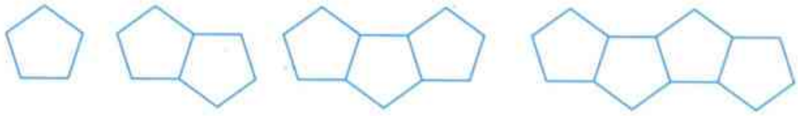
When you put them together in stages to make a fence, you also need joining pieces, so the fence will start to build up as shown below.

a How many pieces of wood would you have in a fence made up in:

i five stages ii n stages iii 45 stages?

b I made a fence out of 124 pieces of wood. How many stages did I use?

5 Regular pentagons of side length 1 cm are joined together to make a pattern, as shown.



1 2 3 4

Copy this pattern and write down the perimeter of each shape. Put the results in a table.

a What is the perimeter of patterns like this made from:

i six pentagons **ii** n pentagons **iii** 50 pentagons?

b What is the largest number of pentagons that can be put together like this to have a perimeter less than 1000 cm?

6 Lamp-posts are put at the end of every 100 m stretch of a motorway, as shown.

1 2 3

a How many lamp-posts are needed for:

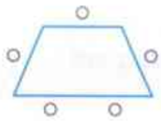
i four 100 m stretches

ii n 100 m stretches

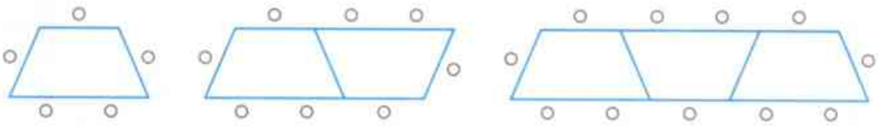
iii 8 km of this motorway.

b A new motorway is being built. The contractor has ordered 1598 lamp-posts. How long is this motorway?

7 A school dining hall had trapezium-shaped tables.
Each table could seat five people, as shown here.



When the tables were joined together, as shown below, the individual tables could not seat as many people.



1 2 3

a In this arrangement, how many could be seated if there were:

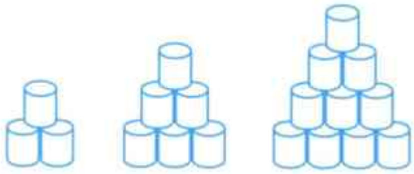
i four tables

ii n tables

iii 13 tables?

b For an outside charity event, up to 200 people had to be seated.
How many tables arranged like this did they need?

8 When setting out tins to make a display of a certain height, you need to know how many tins to start with at the bottom.



a How many tins are needed on the bottom if you wish the display to be:

i five tins high

ii n tins high

iii 18 tins high?

If the display is n tins high, the *total* number of tins is $T = \frac{n(n+1)}{2}$

b Show that this formula gives the correct answer if $n = 2$ or $n = 3$.

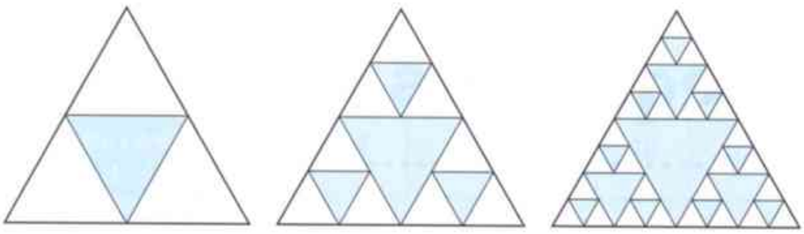
c How many tins will be needed to make a stack 10 tins high?

9 These are the instructions that were used to draw the patterns below.

For pattern 1, draw an equilateral triangle, mark the midpoints of each side and draw and colour in the equilateral triangle formed by these points.

For pattern 2, repeat this with the three white triangles remaining.

For pattern 3, repeat this with the nine white triangles.



Pattern 1 Pattern 2 Pattern 3

The pattern is called a Sierpinski triangle and is one of the earliest examples of a fractal type pattern.

Pattern 4 is completed in the same way.

a Copy and complete the table for patterns 1, 2 and 3.

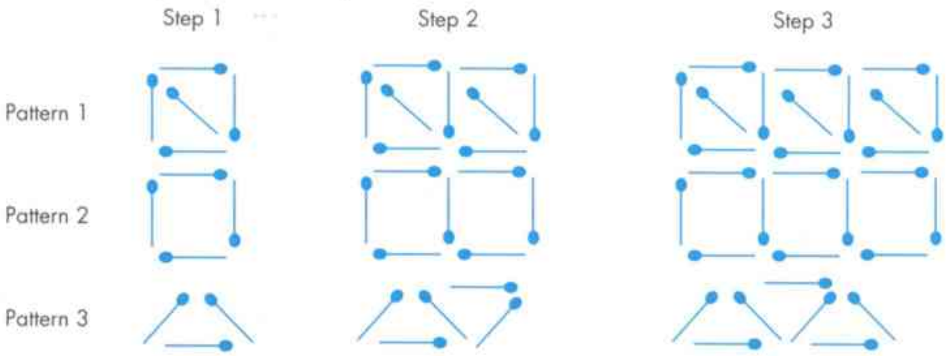
Pattern	1	2	3	4
White triangles	3	9		
Coloured triangles	1	4		
Total	4			

b Use the numbers in the first 3 columns of the table to help you complete column 4. Explain your method.

CORE

10 Thom is building three different patterns with matches.

He builds the patterns in steps.



a Draw step 4 for each pattern.

b Complete this table for pattern 1.

Step	1	2	3	4
Number of matches	5			

c How many matches are needed for step n of pattern 1?

d How many matches are needed for step n of pattern 2?

e How many matches are needed for step n of pattern 3?

f What is the total number of matches needed for step n? Write your answer as simply as possible.

17.4 Further sequences

E

Look at these sequences.

6, 24, 96, 384, 1536, …

48, 24, 12, 6, 3, 1.5, …

In the first, you multiply each term by four to find the next one.

In the second, you multiply each term by 0.5 (or divide it by 2) to find the next term.

The nth term of the first sequence is 1.5×4^n. You can write this as $t_n = 1.5 \times 4^n$.

t_n is mathematical shorthand for the nth term.

Check: The first term is $1.5 \times 4 = 6$ (you can write this as $t_1 = 6$); the third term is $1.5 \times 4^3 = 1.5 \times 64 = 96$ ($t_3 = 6$).

The nth term of the second sequence is 96×0.5^n. You can write this as $t_n = 96 \times 0.5^n$.

Check: The first term is $96 \times 0.5 = 48$ ($t_1 = 48$); the third term is $96 \times 0.5^3 = 96 \times 0.125 = 12$ ($t_3 = 12$).

96×0.5^n can also be written as $\frac{96}{2^n}$. Check that this is correct.

These sequences are called **exponential sequences**. In these sequences, n is an exponent or power.

Example 7

This is the start of an exponential sequence.

11, 12.1, 13.31, 14.641, …

Work out the nth term.

The multiplier from one term to the next is always the same.

$\frac{12.1}{11} = 1.1$ $\frac{13.31}{12.1} = 1.1$ $\frac{14.641}{13.31} = 1.1$

The nth term is $a \times 1.1^n$ where a is a number.

The first term is 11 so $a \times 1.1 = 11 \rightarrow a = \frac{11}{1.1} = 10$.

The nth term is 10×1.1^n.

Sequences may be a combination of different types of sequence.

For example, look at this sequence.

3, 8, 15, 24, 35, …..

It is not linear, because the differences are not all the same.

It is not exponential because the multipliers are not all the same.

Compare the sequence with the square numbers.

n	1	2	3	4	5	…
Sequence	3	8	15	24	35	…
n^2	1	4	9	16	25	…
Difference between sequence and n^2	2	4	6	8	10	…

You can see that the difference is $2n$ each time.

The nth term is $n^2 + 2n$. You can write this as $t_n = n^2 + 2n$.

Check: If $n = 5$ then $n^2 + 2n = 25 + 10 = 35$ which is the fifth term ($t_5 = 35$).

EXERCISE 17E

EXTENDED

1 Find the next term in each of these exponential sequences.

a 12, 36, 144, …
b 13, 39, 117, 351, …
c 2, 10, 50, 250, …
d 20, 24, 28.8, 34.56, …
e 4, 80, 1600, 32000, …
f 240, 120, 60, 30, …
g 1000, 200, 40, 8, …
h 162, 108, 72, 48, …

2 Here are the nth terms of some sequences. In each case, find the first term (t_1) and the fourth term (t_4).

a 6^n
b 8×2^n
c 5×3^n
d 40×0.5^n
e $81 \times \left(\frac{2}{3}\right)^n$

EXTENDED

3 Work out the nth term of each of these sequences and write it in the form $t_n =$.

a 50, 100, 200, 400, 800, …
b 3, 6, 12, 24, 48, …
c 6, 18, 54, 162, ….
d 120, 60, 30, 15, …
e 45, 40.5, 36.45, 32.805, …
f 80, 100, 125, …

4 Selina has $5000 savings. She expects her savings to increase by 40% each year.

Years from now	0	1	2	3	4
Savings ($)	5000	7000	9800		

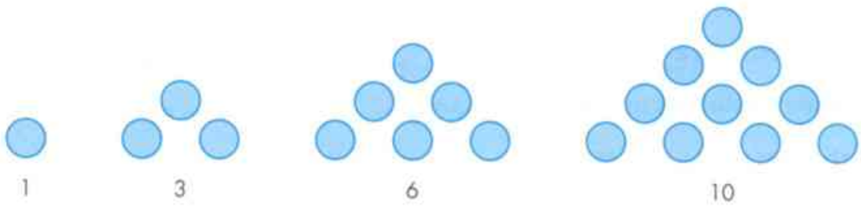
a Work out the two numbers missing from the table.

b Work out an expression for her savings after n years.

5 Find the nth term of each of these sequences.

a 2, 5, 10, 17, 26, …
b 7, 10, 15, 22, 31, …
c 0, 2, 6, 12, 20, …
d 3, 12, 27, 48, 75, …
e 1, 10, 25, 46, 73, …
f 4, 14, 30, 52, 80, …

6 These are the first four triangular numbers.



a Find the next two triangular numbers.

b Find the nth term of this quadratic sequence.

2, 6, 12, 20, 30, …

c Use your answer from part **b** to find an expression for the nth triangular number.

d Work out the 20th triangular number.

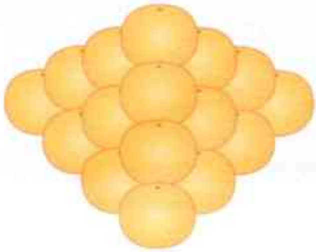
e Show that 1275 is a triangular number.

7 Find the nth term of each of these sequences and write it in the form $t_n =$.

a 0, 7, 26, 63, 124, …
b 51, 58, 77, 114, 157, …
c 2, 10, 30, 68, 130, …
d 4, 14, 36, 76, 140, …

8 Here are some triangles made from oranges.

If these oranges are put in four layers they make a tetrahedron.



a Find the number of oranges in a tetrahedron with four layers.

The number of oranges in a tetrahedron with n layers is $\frac{1}{6}n(n + 1)(n + 2)$.

b Check that this expression gives the correct answer with $n = 4$.

c A shopkeeper has 100 oranges. He wants to display them in a tetrahedron. Work out the largest number of layers he can have.

d Show how to use the formula to add the first 20 triangular numbers.

9 Here is a sequence.

6, 16, 30, 48, 70, …

The nth term of this sequence is given by $t_n = an + bn^2$ where a and b are numbers.

a By looking at the first term, show that $a + b = 6$.

b Use the second term to find another equation involving a and b.

c Solve the equation from parts **a** and **b** simultaneously to find the nth term of the sequence.

Check your progress

Core

- I can continue a given number sequence
- I can recognise patterns in sequences, including a term to term rule
- I can find and use the nth term of a linear sequence
- I can find and use the nth term of a simple quadratic or cubic sequence

Extended

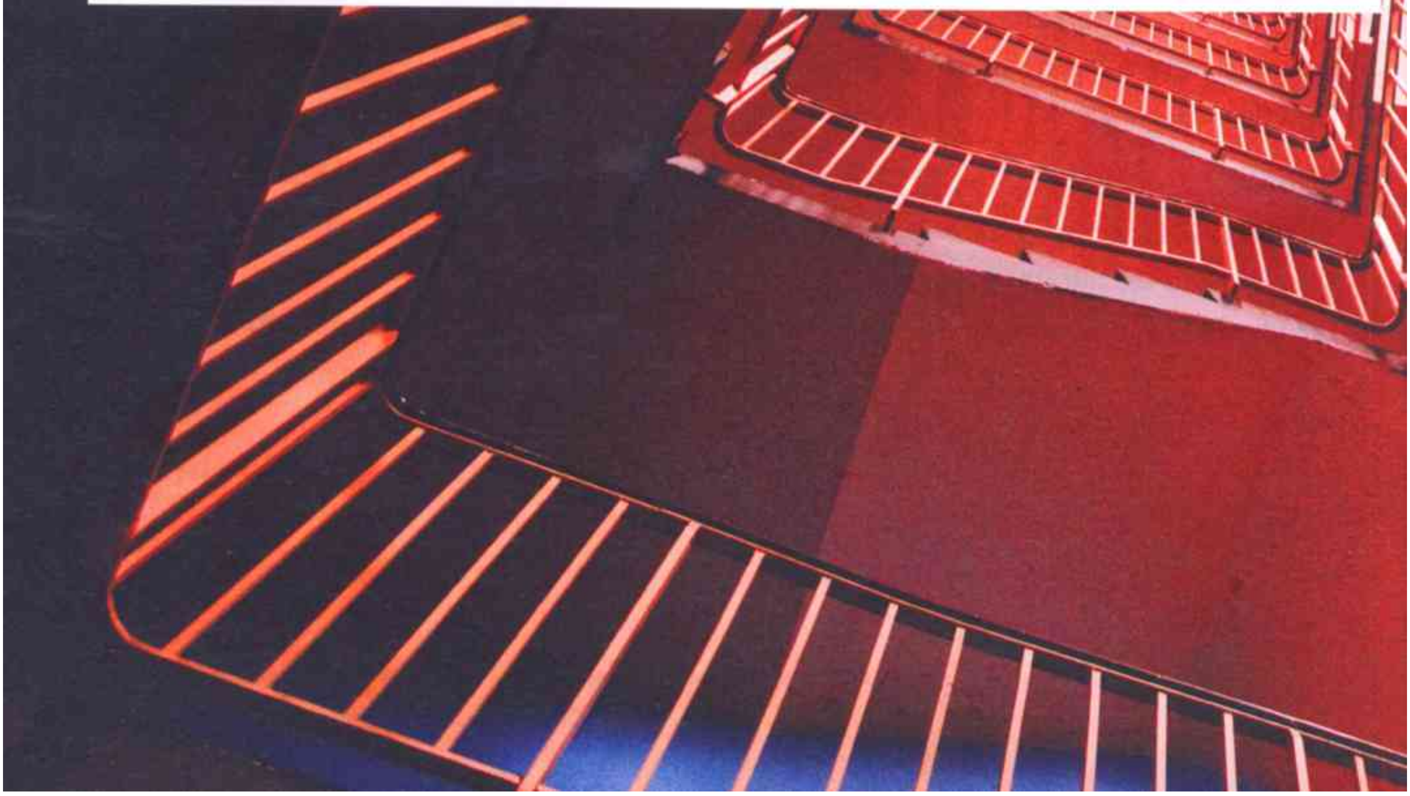
- I understand subscript notation for the terms of a sequence
- I can find and use the nth term of an exponential sequence
- I can find and use the nth term of a simple combination of different types of sequences

Chapter 18
Indices

Topics	Level	Key words
1 Using indices	CORE	index, indices, power, power 1, power 0
2 Negative indices	CORE	negative index, reciprocal
3 Multiplying and dividing with indices	CORE	
4 Fractional indices	EXTENDED	

In this chapter you will learn how to:

CORE	EXTENDED
• Understand the meaning and rules of indices. (C1.7 and E1.7) • Use and interpret positive, negative and zero indices. (C2.4 and E2.4) • Use the rules of indices. (C2.4 and E2.4)	• Use and interpret fractional indices, e.g. solve $32^x = 2$. (E2.4)



Why this chapter matters

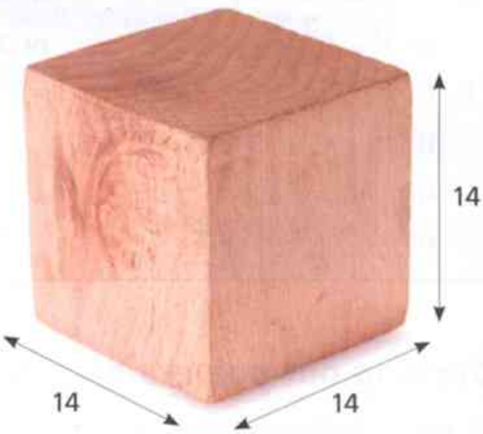
Indices are a useful way to write numbers. They show how different numbers are related to one another and they can make it easier to multiply or divide, or to compare the sizes of different numbers.

You probably already know about powers of numbers from Chapters 5 and 9.

You use find powers of 10 when you write numbers in standard form, such as 3.7×10^6 or 8.92×10^{-5}.

The first is 3 700 000 and the second is 0.000 08 92.

Using powers is a useful 'short cut'.
For example, the centre of the galaxy Andromeda is 24 000 000 000 000 000 000 km from our Sun.
It is much easier to write 24×10^{18}!



You will also find powers in squares and cubes such as 7^2 and 14^3.

Remember that $7^2 = 7 \times 7$ and $14^3 = 14 \times 14 \times 14$. This also saves time in writing out the numbers.

The small number that shows the power is called an index.

- In 7^2 the index is 2 and in 14^3 the index is 3.
- In 3.7×10^6 or 8.92×10^{-5} the indices are 6 and –5.

'Indices' is the plural of index. In the examples above 2, 3, 6 and –5 are indices.

In this chapter you will discover more useful ways to use indices. You might be surprised to find you can write fractions using indices. You can probably see how 8 can be written as a power of two (2^3) but it is not so obvious how $\frac{1}{8}$ can also be written as a power of two.

Warning

The word index has a number of other meanings in English. For example, you will find an index at the back of this book. In this chapter the word is always used to mean a power.

18.1 Using indices

An **index** is a convenient way of writing repetitive multiplications. The plural of index is **indices**.

The index tells you the number of times a number is multiplied by itself. For example:

$4^6 = 4 \times 4 \times 4 \times 4 \times 4 \times 4$ six lots of 4 multiplied together (call this '4 to the **power** 6')

$6^4 = 6 \times 6 \times 6 \times 6$ four lots of 6 multiplied together (call this '6 to the power 4')

$7^3 = 7 \times 7 \times 7$

$12^2 = 12 \times 12$

Example 1

a Write each of these numbers out in full.

i 4^3 **ii** 6^2 **iii** 7^5 **iv** 12^4

b Write each multiplication as a power.

i $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$

ii $13 \times 13 \times 13 \times 13 \times 13$

iii $7 \times 7 \times 7 \times 7$

iv $5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$

---

a i $4^3 = 4 \times 4 \times 4$ **ii** $6^2 = 6 \times 6$

iii $7^5 = 7 \times 7 \times 7 \times 7 \times 7$ **iv** $12^4 = 12 \times 12 \times 12 \times 12$

b i $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^8$

ii $13 \times 13 \times 13 \times 13 \times 13 = 13^5$

iii $7 \times 7 \times 7 \times 7 = 7^4$

iv $5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^7$

Working out powers on your calculator

The power button on your calculator will probably look like this $x^{\square}$.

To work out 5^7 on your calculator use the power key.

5^7 = 5 $x^{\square}$ 7 = 78 125

Two special powers

Power 1

Any number to the power 1 is the same as the number itself. This is always true so normally you do not write the power 1.

For example: $5^1 = 5$ $32^1 = 32$ $(-8)^1 = -8$

Power zero

Any number to the power 0 is equal to 1.

For example:

$5^0 = 1$ $32^0 = 1$ $(-8)^0 = 1$

You can check these results on your calculator.

EXERCISE 18A

CORE

1 Write these expressions using index notation. Do not work them out yet.

a $2 \times 2 \times 2 \times 2$
b $3 \times 3 \times 3 \times 3 \times 3$
c 7×7
d $5 \times 5 \times 5$
e $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$
f $6 \times 6 \times 6 \times 6$
g 4
h $1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1$
i $0.5 \times 0.5 \times 0.5 \times 0.5$
j $100 \times 100 \times 100$

2 Write these power terms out in full. Do not work them out yet.

a 3^4
b 9^3
c 6^2
d 10^5
e 2^{10}
f 8^1
g 0.1^3
h 2.5^2
i 0.7^3
j 1000^2

3 Using the power key on your calculator (or another method), work out the values of the power terms in question **1**.

4 Using the power key on your calculator (or another method), work out the values of the power terms in question **2**.

5 A storage container is in the shape of a cube. The length of the container is 5 m.

To work out the volume of a cube, use the formula:

volume = (length of edge)^3

Work out the total storage space in the container.

6 Write each number as a power of a different number.

The first one has been done for you.

a $32 = 2^5$
b 100
c 8
d 25

7 Without using a calculator, work out the values of these power terms.

a 2^0
b 4^1
c 5^0
d 1^9
e 1^{235}

8 The answers to question **7**, parts **d** and **e**, should tell you something special about powers of 1. What is it?

9 Write the answer to question **1**, part **j** as a power of 10.

10 Write the answer to question **2**, part **j** as a power of 10.

CORE

11 Using your calculator, or otherwise, work out the values of these power terms.

a $(-1)^0$ b $(-1)^1$
c $(-1)^2$ d $(-1)^4$
e $(-1)^5$

12 Using your answers to question **11**, write down the answers to these power terms.

a $(-1)^8$ b $(-1)^{11}$
c $(-1)^{99}$ d $(-1)^{80}$
e $(-1)^{126}$

13 The number 16 777 216 is a power of 2.

It is also a power of 4, a power of 8 and a power of 16.

Write the number as a power of 4, a power of 8 and a power of 16.

18.2 Negative indices

A **negative index** is a convenient way of writing the **reciprocal** of a number or term. (That is, one divided by that number or term.) For example:

$x^{-a} = \frac{1}{x^a}$

Here are some other examples.

$5^{-2} = \frac{1}{5^2}$ $3^{-1} = \frac{1}{3}$ $5x^{-2} = \frac{5}{x^2}$

Example 2

Rewrite each number in the form 2^n.

a 8 **b** $\frac{1}{4}$ **c** -32 **d** $-\frac{1}{64}$

a $8 = 2 \times 2 \times 2 = 2^3$ **b** $\frac{1}{4} = \frac{1}{2^2} = 2^{-2}$

c $-32 = -2^5$ **d** $-\frac{1}{64} = -\frac{1}{2^6} = -2^{-6}$

EXERCISE 18B

CORE

1 Write each of these in fraction form.

a 5^{-3} b 6^{-1} c 10^{-5} d 3^{-2} e 8^{-2}
f 9^{-1} g w^{-2} h t^{-1} i x^{-m} j $4m^{-3}$

2 Write each of these in negative index form.

a $\frac{1}{3^2}$ b $\frac{1}{5}$ c $\frac{1}{10^3}$ d $\frac{1}{m}$ e $\frac{1}{t^n}$

Advice and Tips

If you move a power from top to bottom, or vice versa its sign changes. A negative power means a reciprocal: it does not mean the answer is negative.

3 Change each expression into an index form of the type shown.

a All of the form 2^n

i 16 ii $\frac{1}{2}$

iii $\frac{1}{16}$ iv −8

b All of the form 10^n

i 1000 ii $\frac{1}{10}$

iii $\frac{1}{100}$ iv 1 million

c All of the form 5^n

i 125 ii $\frac{1}{5}$

iii $\frac{1}{25}$ iv $\frac{1}{625}$

d All of the form 3^n

i 9 ii $\frac{1}{27}$

iii $\frac{1}{81}$ iv −243

4 Rewrite each expression in fraction form.

a $5x^{-3}$ b $6r^{-1}$ c $7m^{-2}$

d $4q^{-4}$ e $10y^{-5}$ f $\frac{1}{2}x^{-3}$

g $\frac{1}{2}m^{-1}$ h $\frac{3}{4}t^{-4}$ i $\frac{4}{5}y^{-3}$

j $\frac{7}{8}x^{-5}$

5 Write each fraction in index form.

a $\frac{7}{x^3}$ b $\frac{10}{p}$ c $\frac{5}{t^2}$ d $\frac{8}{m^5}$ e $\frac{3}{y}$

6 Find the value of each number.

a $x = 5$

i x^2 ii x^{-3} iii $4x^{-1}$

b $t = 4$

i t^3 ii t^{-2} iii $5t^{-4}$

c $m = 2$

i m^3 ii m^{-5} iii $9m^{-1}$

d $w = 10$

i w^6 ii w^{-3} iii $25w^{-2}$

7 Two different numbers can be written in the form 2^n.

The sum of the numbers is 40.

What is the difference of the numbers?

CORE

8 x and y are integers.

$x^2 - y^3 = 0$

Work out possible values of x and y.

9 You are given that $8^7 = 2\,097\,152$.

Write down the value of 8^{-7}.

10 Put these in order from smallest to largest:

x^5 x^{-5} x^0

a when x is greater than 1

b when x is between 0 and 1

c when $x = -10$.

11 $M = 3^{-8}$

Write the following in terms of M

a 3^{-9}

b 3^{-7}

c 3^{-5}

18.3 Multiplying and dividing with indices

When you *multiply* powers of the same number or variable, you *add* the indices. For example:

$3^4 \times 3^5 = 3^{(4+5)} = 3^9$

$2^3 \times 2^4 \times 2^5 = 2^{12}$

$10^4 \times 10^{-2} = 10^2$

$10^{-3} \times 10^{-1} = 10^{-4}$

$a^x \times a^y = a^{(x+y)}$

When you *divide* powers of the same number or variable, you *subtract* the indices. For example:

$a^4 \div a^3 = a^{(4-3)} = a^1 = a$

$b^4 \div b^7 = b^{-3}$

$10^4 \div 10^{-2} = 10^6$

$10^{-2} \div 10^{-4} = 10^2$

$a^x \div a^y = a^{(x-y)}$

When you *raise* a power to a further power, you *multiply* the indices. For example:

$(a^2)^3 = a^{2 \times 3} = a^6$

$(a^{-2})^4 = a^{-8}$

$(a^2)^6 = a^{12}$

$(a^x)^y = a^{xy}$

Here are some examples of different kinds of expressions that use powers.

$2a^2 \times 3a^4 = (2 \times 3) \times (a^2 \times a^4)$

$= 6 \times a^6 = 6a^6$

$4a^2b^3 \times 2ab^2 = (4 \times 2) \times (a^2 \times a) \times (b^3 \times b^2)$

$= 8a^3b^5$

$12a^5 \div 3a^2 = (12 \div 3) \times (a^5 \div a^2)$

$= 4a^3$

$(2a^2)^3 = (2)^3 \times (a^2)^3 = 8 \times a^6$

$= 8a^6$

EXERCISE 18C

CORE

1 Write these as single powers of 5.

a $5^2 \times 5^2$ **b** 5×5^2 **c** $5^{-2} \times 5^4$

d $5^6 \times 5^{-3}$ **e** $5^{-2} \times 5^{-3}$

2 Write these as single powers of 6.

a $6^5 \div 6^2$ **b** $6^4 \div 6^4$ **c** $6^4 \div 6^{-2}$

d $6^{-3} \div 6^4$ **e** $6^{-3} \div 6^{-5}$

3 Simplify these and write them as single powers of a.

a $a^2 \times a$ **b** $a^3 \times a^2$ **c** $a^4 \times a^3$

d $a^6 \div a^2$ **e** $a^3 \div a$ **f** $a^5 \div a^4$

4 **a** $a^x \times a^y = a^{10}$

Write down a possible pair of values of x and y.

b $a^x \div a^y = a^{10}$

Write down a possible pair of values of x and y.

5 Write these as single powers of 4.

a $(4^2)^3$ **b** $(4^3)^5$ **c** $(4^1)^6$

d $(4^3)^{-2}$ **e** $(4^{-2})^{-3}$ **f** $(4^7)^0$

CORE

6 Simplify these expressions.

a $2a^2 \times 3a^3$ b $3a^4 \times 3a^{-2}$

c $(2a^2)^3$ d $-2a^2 \times 3a^2$

e $-4a^3 \times -2a^5$ f $-2a^4 \times 5a^{-7}$

7 Simplify these expressions.

a $6a^3 \div 2a^2$ b $12a^5 \div 3a^2$

c $15a^5 \div 5a$ d $18a^{-2} \div 3a^{-1}$

e $24a^5 \div 6a^{-2}$ f $30a \div 6a^5$

Advice and Tips

Deal with numbers and indices separately and do not confuse the rules.
For example: $12a^5 \div 4a^2$
$= (12 \div 4) \times (a^5 \div a^2)$

8 Simplify these expressions.

a $2a^2b^3 \times 4a^3b$

b $5a^2b^4 \times 2ab^{-3}$

c $6a^2b^3 \times 5a^{-4}b^{-5}$

d $12a^2b^4 \div 6ab$

e $24a^{-3}b^4 \div 3a^2b^{-3}$

9 Simplify these expressions.

a $\dfrac{6a^4b^3}{2ab}$

b $\dfrac{2a^2bc^2 \times 6abc^3}{4ab^2c}$

c $\dfrac{3abc \times 4a^3b^2c \times 6c^2}{9a^2bc}$

10 Write down **two** possible:

a multiplication questions with an answer of $12x^2y^5$

b division questions with an answer of $12x^2y^5$.

11 a, b and c are three different positive integers.

What is the smallest possible value of a^2b^3c?

12 $8^{12} = A$

Find the following in terms of A

a 8^{24} b 8^{-12}

c 8^6 d 2^{12}

13 $x^n = y$

a Show that $x^{2n+1} = xy^2$

b Find a similar expression for x^{2n-1}

18.4 Fractional indices

E

Indices of the form $\frac{1}{n}$

Consider the problem $7^x \times 7^x = 7$. This can be written as:

$$7^{(x+x)} = 7$$

$$7^{2x} = 7^1 \Rightarrow 2x = 1 \Rightarrow x = \tfrac{1}{2}$$

If you now substitute $x = \frac{1}{2}$ back into the original equation, you see that:

$$7^{\frac{1}{2}} \times 7^{\frac{1}{2}} = 7$$

This makes $7^{\frac{1}{2}}$ the same as $\sqrt{7}$.

You can similarly show that $7^{\frac{1}{3}}$ is the same as $\sqrt[3]{7}$. And that, generally:

$$x^{\frac{1}{n}} = \sqrt[n]{x} \text{ (}n\text{th root of } x\text{)}$$

So in summary:

Power $\frac{1}{2}$ is the same as positive square root.

Power $\frac{1}{3}$ is the same as cube root.

Power $\frac{1}{n}$ is the same as nth root.

For example:

$$49^{\frac{1}{2}} = \sqrt{49} = 7$$

$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$$

$$10\,000^{\frac{1}{4}} = \sqrt[4]{10\,000} = 10$$

$$36^{-\frac{1}{2}} = \frac{1}{\sqrt{36}} = \frac{1}{6}$$

If you have an expression in the form $\left(\frac{a}{b}\right)^n$ you can calculate it as $\frac{a^n}{b^n}$ and then write it as a fraction.

Example 3

Write $\left(\frac{16}{25}\right)^{\frac{1}{2}}$ as a fraction.

You can find the power of the numerator and denominator separately.

$$\left(\frac{16}{25}\right)^{\frac{1}{2}} = \frac{16^{\frac{1}{2}}}{25^{\frac{1}{2}}}$$

$$= \frac{4}{5}$$

EXERCISE 18D

EXTENDED

1 Evaluate each number.

a $25^{\frac{1}{2}}$	**b** $100^{\frac{1}{2}}$	**c** $64^{\frac{1}{2}}$	**d** $81^{\frac{1}{2}}$	**e** $625^{\frac{1}{2}}$
f $27^{\frac{1}{3}}$	**g** $64^{\frac{1}{3}}$	**h** $1000^{\frac{1}{3}}$	**i** $125^{\frac{1}{3}}$	**j** $512^{\frac{1}{3}}$
k $144^{\frac{1}{2}}$	**l** $400^{\frac{1}{2}}$	**m** $625^{\frac{1}{4}}$	**n** $81^{\frac{1}{4}}$	**o** $100\,000^{\frac{1}{5}}$
p $729^{\frac{1}{6}}$	**q** $32^{\frac{1}{5}}$	**r** $1024^{\frac{1}{10}}$	**s** $1296^{\frac{1}{4}}$	**t** $216^{\frac{1}{3}}$
u $16^{-\frac{1}{2}}$	**v** $8^{-\frac{1}{3}}$	**w** $81^{-\frac{1}{4}}$	**x** $3125^{-\frac{1}{5}}$	**y** $1000\,000^{-\frac{1}{6}}$

2 Evaluate each number.

a $\left(\frac{25}{36}\right)^{\frac{1}{2}}$	**b** $\left(\frac{100}{36}\right)^{\frac{1}{2}}$	**c** $\left(\frac{64}{81}\right)^{\frac{1}{2}}$	**d** $\left(\frac{81}{25}\right)^{\frac{1}{2}}$
e $\left(\frac{25}{64}\right)^{\frac{1}{2}}$	**f** $\left(\frac{27}{125}\right)^{\frac{1}{3}}$	**g** $\left(\frac{8}{512}\right)^{\frac{1}{3}}$	**h** $\left(\frac{1000}{64}\right)^{\frac{1}{3}}$
i $\left(\frac{64}{125}\right)^{\frac{1}{3}}$	**j** $\left(\frac{512}{343}\right)^{\frac{1}{3}}$		

3 Use the general rule for raising a power to another power to prove that $x^{\frac{1}{n}}$ is equivalent to $\sqrt[n]{x}$.

4 Which of these is the odd one out?

$16^{-\frac{1}{4}}$ $64^{-\frac{1}{2}}$ $8^{-\frac{1}{3}}$

Show how you decided.

5 Imagine that you are the teacher.

Write down how you would teach the class that $27^{-\frac{1}{3}}$ is equal to $\frac{1}{3}$.

6 $x^{-\frac{2}{3}} = y^{\frac{1}{3}}$

Find values for x and y that make this equation work.

7 Solve these equations.

a $2^x = 8$	**b** $8^x = 2$	**c** $4^x = 1$	**d** $16^x = 4$
e $100^x = 10$	**f** $81^x = 3$	**g** $16^x = 2$	**h** $125^x = 5$
i $1000^x = 10$	**j** $400^x = 20$	**k** $512^x = 8$	**l** $128^x = 2$

Indices of the form $\frac{a}{b}$

Here are two examples of this form.

$$t^{\frac{2}{3}} = t^{\frac{1}{3}} \times t^{\frac{1}{3}} = (\sqrt[3]{t})^2 \quad 81^{\frac{3}{4}} = (\sqrt[4]{81})^3 = 3^3 = 27$$

If you have an expression of the form $\left(\frac{a}{b}\right)^{-n}$ you can invert it to calculate it as a fraction.

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$$

Example 4

Evaluate each number. **a** $16^{-\frac{1}{4}}$ **b** $32^{-\frac{4}{5}}$

When dealing with the negative index remember that it means reciprocal.

Work through problems like these one step at a time.

Step 1: Rewrite the calculation as a fraction by dealing with the negative power.

Step 2: Take the root of the base number given by the denominator of the fraction.

Step 3: Raise the result to the power given by the numerator of the fraction.

Step 4: Write out the answer as a fraction.

a **Step 1:** $16^{-\frac{1}{4}} = \left(\frac{1}{16}\right)^{\frac{1}{4}}$ **Step 2:** $16^{\frac{1}{4}} = \sqrt[4]{16} = 2$ **Step 3:** $2^1 = 2$ **Step 4:** $16^{-\frac{1}{4}} = \frac{1}{2}$

b **Step 1:** $32^{-\frac{4}{5}} = \left(\frac{1}{32}\right)^{\frac{4}{5}}$ **Step 2:** $32^{\frac{1}{5}} = \sqrt[5]{32} = 2$ **Step 3:** $2^4 = 16$ **Step 4:** $32^{-\frac{4}{5}} = \frac{1}{16}$

Example 5

Write $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$ as a fraction.

$$\left(\frac{8}{27}\right)^{-\frac{2}{3}} = \left(\frac{27}{8}\right)^{\frac{2}{3}}$$

$$= \frac{27^{\frac{2}{3}}}{8^{\frac{2}{3}}}$$

$$= \frac{(\sqrt[3]{27})^2}{(\sqrt[3]{8})^2} = \frac{3^2}{2^2} = \frac{9}{4}$$

EXERCISE 18E

EXTENDED

1 Evaluate these.

a $32^{\frac{4}{5}}$ b $125^{\frac{2}{3}}$

c $1296^{\frac{3}{4}}$ d $243^{\frac{4}{5}}$

2 Rewrite these in index form.

a $\sqrt[3]{t^2}$ b $\sqrt[4]{m^3}$

c $\sqrt[5]{k^2}$ d $\sqrt{x^3}$

3 Evaluate these.

a $8^{\frac{2}{3}}$ b $27^{\frac{2}{3}}$

c $16^{\frac{3}{2}}$ d $625^{\frac{5}{4}}$

EXTENDED

4 Evaluate these.

a $25^{-\frac{1}{2}}$ b $36^{-\frac{1}{2}}$

c $16^{-\frac{1}{4}}$ d $81^{-\frac{1}{4}}$

e $16^{-\frac{1}{2}}$ f $8^{-\frac{1}{3}}$

g $32^{-\frac{1}{5}}$ h $27^{-\frac{1}{3}}$

5 Evaluate these.

a $25^{-\frac{3}{2}}$ b $36^{-\frac{3}{2}}$

c $16^{-\frac{3}{4}}$ d $81^{-\frac{3}{4}}$

e $64^{-\frac{4}{3}}$ f $8^{-\frac{2}{3}}$

g $32^{-\frac{2}{5}}$ h $27^{-\frac{2}{3}}$

6 Evaluate these.

a $100^{-\frac{5}{2}}$ b $144^{-\frac{1}{2}}$

c $125^{-\frac{2}{3}}$ d $9^{-\frac{3}{2}}$

e $4^{-\frac{5}{2}}$ f $64^{-\frac{5}{6}}$

g $27^{-\frac{4}{3}}$ h $169^{-\frac{1}{2}}$

7 Which of these is the odd one out?

$16^{-\frac{3}{4}}$ $64^{-\frac{1}{2}}$ $8^{-\frac{2}{3}}$

Show how you decided.

8 Imagine that you are the teacher.

Write down how you would teach the class that $27^{-\frac{2}{3}}$ is equal to $\frac{1}{9}$.

9 Write these as fractions.

a $\left(\frac{9}{4}\right)^{\frac{3}{2}}$ b $\left(\frac{27}{125}\right)^{\frac{2}{3}}$

c $\left(\frac{16}{9}\right)^{\frac{5}{2}}$ d $\left(\frac{4}{49}\right)^{\frac{3}{2}}$

e $\left(\frac{64}{27}\right)^{\frac{2}{3}}$ f $\left(\frac{16}{81}\right)^{\frac{3}{4}}$

g $\left(\frac{125}{64}\right)^{\frac{4}{3}}$ h $\left(\frac{64}{729}\right)^{\frac{5}{6}}$

10 Write these as fractions.

a $\left(\frac{3}{5}\right)^{-2}$ b $\left(\frac{4}{3}\right)^{-3}$

c $\left(\frac{9}{5}\right)^{-3}$ d $\left(\frac{2}{3}\right)^{-5}$

e $\left(\frac{9}{4}\right)^{-\frac{3}{2}}$ f $\left(\frac{4}{9}\right)^{-\frac{5}{2}}$

g $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$ h $\left(\frac{49}{25}\right)^{-\frac{3}{2}}$

i $\left(\frac{125}{64}\right)^{-\frac{2}{3}}$ j $\left(\frac{25}{64}\right)^{-\frac{3}{2}}$

k $\left(\frac{16}{81}\right)^{-\frac{5}{4}}$ l $\left(\frac{2187}{128}\right)^{-\frac{5}{7}}$

11 Simplify these expressions.

a $x^{\frac{3}{2}} \times x^{\frac{5}{2}}$ b $x^{\frac{1}{2}} \times x^{-\frac{3}{2}}$

c $(8y^3)^{\frac{2}{3}}$ d $5x^{\frac{3}{2}} \div \frac{1}{2}x^{-\frac{1}{2}}$

e $4x^{\frac{1}{2}} \times 5x^{-\frac{3}{2}}$ f $\left(\frac{27}{y^3}\right)^{-\frac{1}{3}}$

12 Simplify these.

a $x^{\frac{1}{2}} \times x^{\frac{1}{2}}$ b $d^{-\frac{1}{2}} \times d^{-\frac{1}{2}}$ c $t^{\frac{1}{2}} \times t$

d $(x^{\frac{1}{2}})^4$ e $(y^2)^{\frac{1}{4}}$ f $a^{\frac{1}{2}} \times a^{\frac{3}{2}} \times a^2$

13 Simplify these.

a $x \div x^{\frac{1}{2}}$ b $y^{\frac{1}{2}} \div y^{1\frac{1}{2}}$ c $a^{\frac{1}{3}} \times a^{\frac{4}{3}}$

d $t^{-\frac{1}{2}} \times t^{-\frac{3}{2}}$ e $\frac{1}{d^{-2}}$ f $\frac{k^{\frac{1}{2}} \times k^{\frac{3}{2}}}{k^2}$

14 $x^{\frac{2}{3}} = y$

Find $x^{\frac{3}{2}}$ in terms of y.

Check your progress

Core

- I understand the meaning of fractional, negative and zero indices
- I can use the rules of indices to simplify numerical expressions such as $2^{-3} \times 2^4$ or $(2^3)^2$

Chapter 19

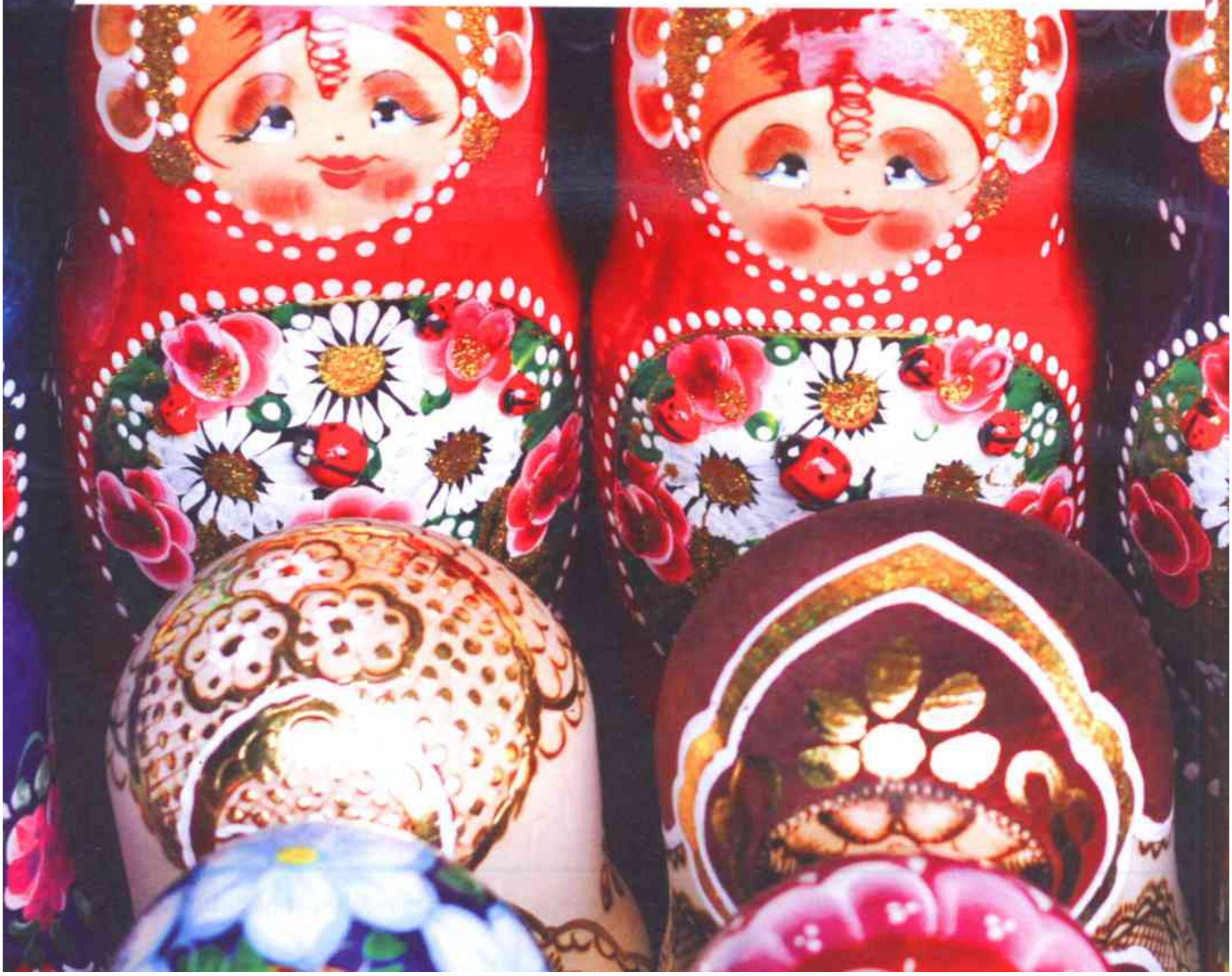
Proportion

Topics	Level	Key words
1 Direct proportion	EXTENDED	direct proportion, constant of proportionality
2 Inverse proportion	EXTENDED	inverse proportion

In this chapter you will learn how to:

EXTENDED

- Express direct and inverse proportion in algebraic terms and use this form of expression to find unknown quantities. (E2.8)



Why this chapter matters

In many real-life situations, variables are connected by a rule or relationship. It may be that as one variable increases the other increases. Alternatively, it may be that as one variable increases the other decreases.

In this chapter you will look at how quantities vary when they are related in some way.

As this plant gets older it becomes taller.

As the storm increases the number of sunbathers decreases.

As this car gets older it is worth less (and eventually it is worthless!).

As more songs are downloaded, there is less money left on the voucher.

Try to think of other variables that are connected in this way.

19.1 Direct proportion

E

There is direct proportion between two variables when one variable is a simple multiple of the other. That is, their ratio is a constant.

For example:

1 kilogram = 2.2 pounds	There is a multiplying factor of 2.2 between kilograms and pounds.
Area of a circle = πr^2	There is a multiplying factor of π between the area of a circle and the square of its radius.

Any question involving direct proportion usually requires you first to find this multiplying factor (called the **constant of proportionality**), then to use it to solve a problem.

The symbol for proportion is $\propto$.

So the statement 'Pay is directly proportional to time' can be mathematically written as:

pay $\propto$ *time*

which implies that:

pay = k × *time*

where k is the constant of proportionality.

There are four steps to be followed when you are using proportionality to solve problems.

Step 1: Set up the statement, using the proportionality symbol (you may use symbols to represent the variables).

Step 2: Set up the equation, using a constant of proportionality.

Step 3: Use given information to work out the value of the constant of proportionality.

Step 4: Substitute the value of the constant of proportionality into the equation and use this equation to find unknown values.

Example 1

The cost of an item is directly proportional to the time spent making it. An item taking 6 hours to make costs $30. Find:

a the cost of an item that takes 5 hours to make

b the length of time it takes to make an item costing $40.

Step 1: Let C dollars be the cost of making an item and t hours the time it takes.

$C \propto t$

Step 2: Setting up the equation gives:

$C = kt$

where k is the constant of proportionality.

Note that you can 'replace' the proportionality sign $\propto$ with $= k$ to obtain the proportionality equation.

Step 3: Since $C = 30$ when $t = 6$, then $30 = 6k$

$\Rightarrow \frac{30}{6} = k$

$\Rightarrow k = 5$

Step 4: So the formula is $C = 5t$.

a When $t = 5$ $\quad C = 5 \times 5 = 25$

So the cost is \$25.

b When $C = 40$ $\quad 40 = 5 \times t$

$\Rightarrow \frac{40}{5} = t \Rightarrow t = 8$

So the time spent making the item is 8 hours.

EXERCISE 19A

For questions **1** to **4**, first find the value of k, the constant of proportionality, and then the formula connecting the variables.

1 T is directly proportional to M. If $T = 20$ when $M = 4$, find the value of:

a T when $M = 3$

b M when $T = 10$.

2 W is directly proportional to F. If $W = 45$ when $F = 3$, find the value of:

a W when $F = 5$

b F when $W = 90$.

3 Q varies directly with P. If $Q = 100$ when $P = 2$, find the value of:

a Q when $P = 3$

b P when $Q = 300$.

4 X varies directly with Y. If $X = 17.5$ when $Y = 7$, find the value of:

a X when $Y = 9$

b Y when $X = 30$.

5 The distance covered by a train is directly proportional to the time taken for the journey. The train travels 105 kilometres in 3 hours.

a What distance will the train cover in 5 hours?

b How much time will it take for the train to cover 280 kilometres?

EXTENDED

6 The cost of fuel delivered to your door is directly proportional to the mass received. When 250 kg is delivered, it costs 47.50 dollars.

a How much will it cost to have 350 kg delivered?

b How much would be delivered if the cost were 33.25 dollars?

7 The number of children who can play safely in a playground is directly proportional to the area of the playground. A playground with an area of 210 m^2 is safe for 60 children.

a How many children can safely play in a playground of area 154 m^2?

b A playgroup has 24 children. What is the smallest playground area in which they could safely play?

8 The number of spaces in a car park is directly proportional to the area of the car park.

a A car park has 300 parking spaces in an area of 4500 m^2.

It is decided to increase the area of the car park by 500 m^2 to make extra spaces.

How many extra spaces will be made?

b The old part of the car park is redesigned so that the original area has 10% more parking spaces.

How many more spaces than in the original car park will there be altogether if the number of spaces in the new area is directly proportional to the number in the redesigned car park?

9 The number of passengers in a bus queue is directly proportional to the time that the person at the front of the queue has spent waiting.

Katya is the first to arrive at a bus stop. When she has been waiting 5 minutes the queue has 20 passengers.

A bus has room for 70 passengers.

How long had Katya been in the queue if the bus fills up from empty when it arrives and all passengers get on?

Direct proportions involving squares, cubes, square roots and cube roots

The process is the same as for a linear direct variation, as the next example shows.

Example 2

The cost of a circular badge is directly proportional to the square of its radius. The cost of a badge with a radius of 2 cm is \$0.68. Find:

a the cost of a badge of radius 2.4 cm **b** the radius of a badge costing \$1.53.

Step 1: Let C be the cost in dollars and r the radius of a badge in centimetres.

$C \propto r^2$

Step 2: Setting up the equation gives:

$C = kr^2$

where k is the constant of proportionality.

Step 3: $C = 0.68$ when $r = 2$. So:

$0.68 = 4k$

$\Rightarrow \frac{0.68}{4} = k \Rightarrow k = 0.17$

Step 4: So the formula is $C = 0.17r^2$.

a When $r = 2.4$ $\quad C = 0.17 \times 2.4^2 = 0.98$ to 2 decimal places.

Rounding gives the cost as \$0.98.

b When $C = 1.53$ $\quad 1.53 = 0.17r^2$

$\Rightarrow \frac{1.53}{0.17} = 9 = r^2$

$\Rightarrow r = \sqrt{9} = 3$

Hence, the radius is 3 cm.

EXERCISE 19B

For questions **1** to **6**, first find k, the constant of proportionality, and then the formula connecting the variables.

1 T is directly proportional to x^2. If $T = 36$ when $x = 3$, find the value of:

a T when $x = 5$ **b** x when $T = 400$.

2 W is directly proportional to M^2. If $W = 12$ when $M = 2$, find the value of:

a W when $M = 3$ **b** M when $W = 75$.

3 E varies directly with $\sqrt{C}$. If $E = 40$ when $C = 25$, find the value of:

a E when $C = 49$ **b** C when $E = 10.4$.

4 X is directly proportional to $\sqrt{Y}$. If $X = 128$ when $Y = 16$, find the value of:

a X when $Y = 36$ **b** Y when $X = 48$.

5 P is directly proportional to f^3. If $P = 400$ when $f = 10$, find the value of:

a P when $f = 4$ **b** f when $P = 50$.

6 y is directly proportional to $\sqrt[3]{x}$. If $y = 100$ when $x = 125$, find the value of:

a y when $x = 64$ **b** x when $y = 40$.

7 The cost of serving tea and biscuits varies directly with the square root of the number of people at the buffet. It costs \$25 to serve tea and biscuits to 100 people.

a How much will it cost to serve tea and biscuits to 400 people?

b For a cost of \$37.50, how many people could be served tea and biscuits?

EXTENDED

EXTENDED

8 In an experiment, the temperature, in °C, varied directly with the square of the pressure, in atmospheres (atm). The temperature was 20 °C when the pressure was 5 atm.

a What will the temperature be at 2 atm?

b What will the pressure be at 80 °C?

9 The mass, in grams, of ball bearings varies directly with the cube of the radius, measured in millimetres. A ball bearing of radius 4 mm has a mass of 115.2 g.

a What will be the mass of a ball bearing of radius 6 mm?

b A ball bearing has a mass of 48.6 g. What is its radius?

10 The energy, in joules (J), of a particle varies directly with the square of its speed, in m/s. A particle moving at 20 m/s has 50 J of energy.

a How much energy has a particle moving at 4 m/s?

b At what speed is a particle moving if it has 200 J of energy?

11 The cost, in dollars, of a trip varies directly with the square root of the number of miles travelled. The cost of a 100-mile trip is 35 dollars.

a What is the cost of a 500-mile trip (to the nearest dollar)?

b What is the distance of a trip costing 70 dollars?

12 A sculptor is making statues.

The amount of clay used is directly proportional to the cube of the height of the statue.

A statue is 10 cm tall and uses 500 cm³ of clay.

How much clay will a similar statue use if it is twice as tall?

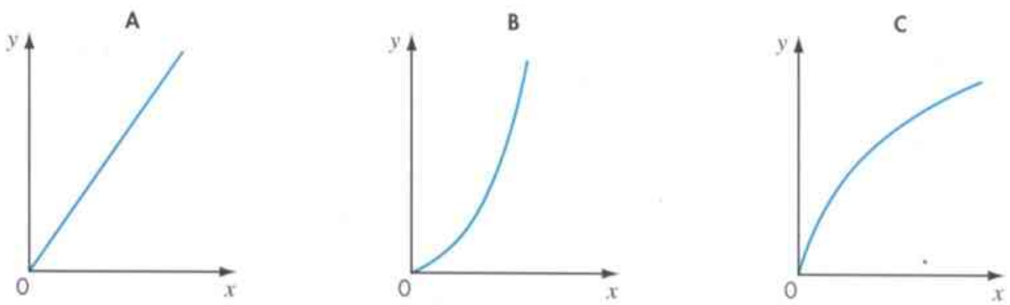
13 The cost of making different-sized machines is proportional to the time taken.

A small machine costs $100 and takes two hours to make.

How much will a large machine cost that takes 5 hours to build?

14 The sketch graphs show each of these proportion statements.

a $y \propto x^2$ **b** $y \propto x$ **c** $y \propto \sqrt{x}$



Match each statement to the correct sketch.

15 Here are two tables.

Match each table to a graph in question **14**.

a

x	1	2	3
y	3	12	27

b

x	1	2	3
y	3	6	9

19.2 Inverse proportion

E

There is inverse proportion between two variables when one variable is directly proportional to the *reciprocal* of the other. That is, the product of the two variables is constant. So, as one variable increases, the other decreases.

For example, the faster you travel over a given distance, the less time it takes. So there is an inverse variation between speed and time. Speed is inversely proportional to time.

$S \propto \frac{1}{T}$ and so $S = \frac{k}{T}$

which can be written as $ST = k$.

Example 3

M is inversely proportional to R. If $M = 9$ when $R = 4$, find the value of:

a M when $R = 2$ **b** R when $M = 3$.

Step 1: $M \propto \frac{1}{R}$

Step 2: Setting up the equation gives:

$M = \frac{k}{R}$

where k is the **constant of proportionality.**

Step 3: $M = 9$ when $R = 4$. So $9 = \frac{k}{4}$

$\Rightarrow 9 \times 4 = k \Rightarrow k = 36$

Step 4: The formula is $M = \frac{36}{R}$

a When $R = 2$, then $M = \frac{36}{2} = 18$

b When $M = 3$, then $3 = \frac{36}{R} \Rightarrow 3R = 36 \Rightarrow R = 12$

EXERCISE 19C

EXTENDED

For questions **1** to **6**, first find the formula connecting the variables.

1 T is inversely proportional to m. If $T = 6$ when $m = 2$, find the value of:

a T when $m = 4$ **b** m when $T = 4.8$.

2 W is inversely proportional to x. If $W = 5$ when $x = 12$, find the value of:

a W when $x = 3$ **b** x when $W = 10$.

3 Q varies inversely with $(5 - t)$. If $Q = 8$ when $t = 3$, find the value of:

a Q when $t = 10$ **b** t when $Q = 16$.

4 M varies inversely with t^2. If $M = 9$ when $t = 2$, find the value of:

a M when $t = 3$ **b** t when $M = 1.44$.

5 W is inversely proportional to $\sqrt{T}$. If $W = 6$ when $T = 16$, find the value of:

a W when $T = 25$ **b** T when $W = 2.4$.

6 y is inversely proportional to the cube of x. If $y = 4$ when $x = 2$, find the value of:

a y when $x = 1$ **b** x when $y = \frac{1}{2}$.

7 The grant available to a group of students was inversely proportional to the number of students. When 30 students needed a grant, they received \$60 each.

a What would the grant have been if 120 students had needed one?

b If the grant had been \$50 each, how many students would have received it?

8 While doing underwater tests in an ocean, scientists noticed that the temperature, in °C, was inversely proportional to the depth, in kilometres. When the temperature was 6 °C, the scientists were at a depth of 4 km.

a What would the temperature have been at a depth of 8 km?

b At what depth would they find the temperature at 2 °C?

9 A new engine had serious problems. The distance it went, in kilometres, without breaking down was inversely proportional to the square of its speed in metres per second (m/s). When the speed was 12 m/s, the engine lasted 3 km.

a Find the distance covered before a breakdown, when the speed is 15 m/s.

b On one test, the engine broke down after 6.75 km. What was the speed?

10 In a balloon it was noticed that the pressure, in atmospheres (atm), was inversely proportional to the square root of the height, in metres. When the balloon was at a height of 25 m, the pressure was 1.44 atm.

a What was the pressure at a height of 9 m?

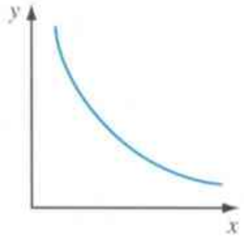
b What would the height have been if the pressure was 0.72 atm?

11 The amount of waste from a firm, measured in tonnes per hour, is inversely proportional to the square root of the area of the filter beds that clean it, in square metres (m^2). The firm produces 1.25 tonnes of waste per hour, with filter beds of size 0.16 m^2.

a The filter beds used to be only 0.01 m^2. How much waste did the firm produce then?

b How much waste could be produced if the filter beds were 0.75 m^2?

12 Which statement is represented by the graph? Give a reason for your answer.



A $y \propto x$

B $y \propto \frac{1}{x}$

C $y \propto \sqrt{x}$

13 In the table, y is inversely proportional to the cube root of x.

Complete the table, leaving your answers as fractions.

x	8	27	
y	1		$\frac{1}{2}$

Check your progress

Extended

- I understand the terms direct and inverse proportion
- I can express direct and inverse proportion in algebraic terms
- I can use this form of expression to find unknown quantities

Chapter 20

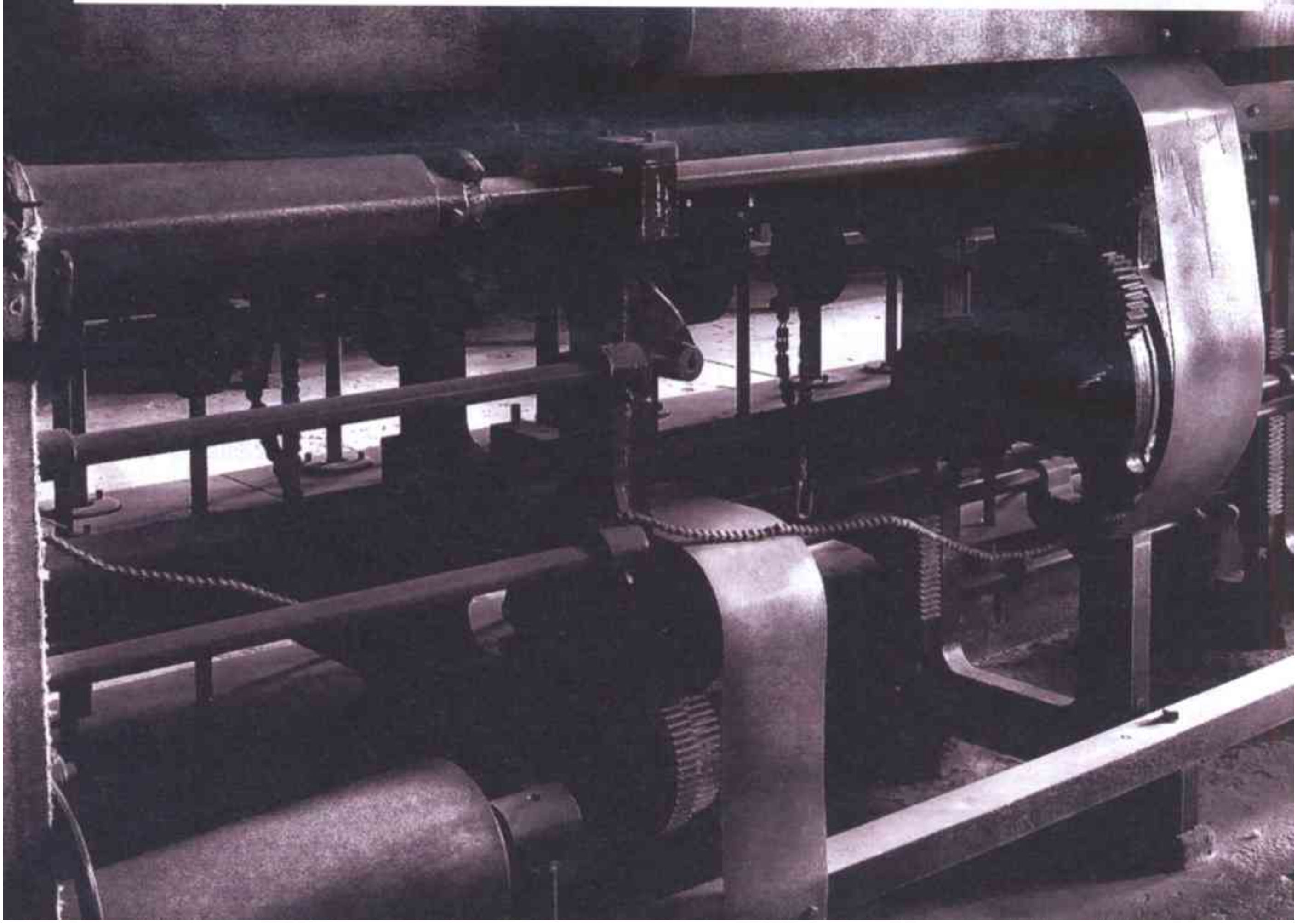
Linear programming

Topics	Level	Key words
1 Graphical inequalities	EXTENDED	boundary, region, included, origin, dashed line, solid line
2 More than one inequality	EXTENDED	required
3 Linear programming	EXTENDED	linear programming

In this chapter you will learn how to:

EXTENDED

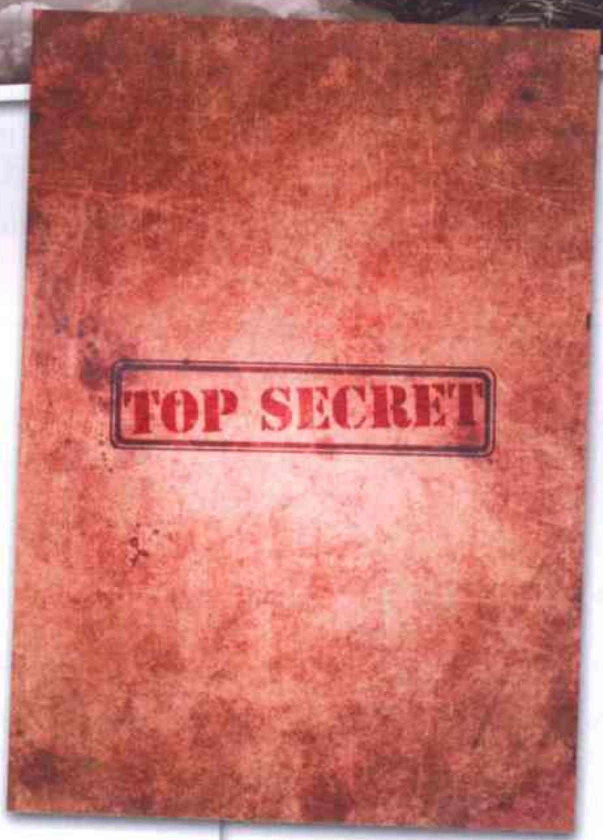
- Represent inequalities graphically. (E2.6)
- Use this representation in the solution of simple linear programming problems. (E2.6)



Why this chapter matters

The theory of linear programming has been used by many companies to reduce their costs and increase productivity.

The theory of linear programming was developed at the start of the Second World War in 1939.

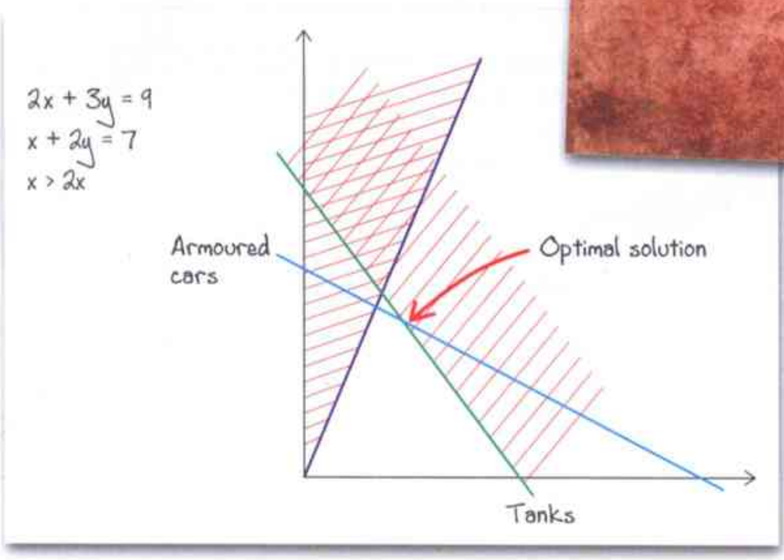


It was used to work out ways to supply armaments as efficiently as possible. It was such a powerful tool that the British and Americans did not want the Germans to know about it, so it was not made public until 1947.

George Dantzig was one of the inventors of linear programming. He came late to a lecture at University one day and saw two problems written on the blackboard. He copied them, thinking they were the homework assignment.

He solved both problems, but apologised to the lecturer later as he found them a little harder than the usual homework, so he took a few days to solve them and was late handing them in.

The lecturer was astonished. The problems he had written on the board were not homework but examples of 'impossible problems'. Not any more!



20.1 Graphical inequalities

E

A linear inequality can be plotted on a graph. The result is a **region** that lies on one side or the other of a straight line. You will recognise an inequality by the fact that it looks like an equation but instead of the equals sign it has an inequality sign: $<$, $>$, $\leqslant$, or $\geqslant$.

The following are examples of linear inequalities that can be represented on a graph.

$y < 3$ $\quad x > 7$ $\quad -3 \leqslant y < 5$ $\quad y \geqslant 2x + 3$ $\quad 2x + 3y < 6$ $\quad y \leqslant x$

The method for graphing an inequality is to draw the **boundary** line that defines the inequality. This is found by replacing the inequality sign with an equals sign. When a strict inequality is stated ($<$ or $>$), the boundary line should be drawn as a *dashed* line to show that it is not included in the range of values. When $\leqslant$ or $\geqslant$ is used to state the inequality, the boundary line should be drawn as a *solid* line to show that the boundary is **included**.

After the boundary line has been drawn, shade the *unwanted region*.

To confirm on which side of the line the region lies, choose any point that is not on the boundary line and test it in the inequality. If it satisfies the inequality, that is the side required. If it doesn't, the other side is required.

Work through the six inequalities in this example to see how the procedure is applied.

Advice and Tips

You should shade the unwanted region of the graph, leaving the region that satisfies the inequality unshaded.

Example 1

Show each inequality on a graph.

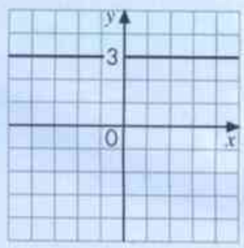
a $y \leqslant 3$ **b** $x > 7$ **c** $-3 \leqslant y < 5$

d $y \leqslant 2x + 3$ **e** $2x + 3y < 6$ **f** $y \leqslant x$

a Draw the line $y = 3$. Since the inequality is stated as $\leqslant$, the line is *solid*. Test a point that is not on the line. The **origin** is always a good choice if possible, as 0 is easy to test.

Putting 0 into the inequality gives $0 \leqslant 3$. The inequality is satisfied and so the region containing the origin is the side we want.

Shade the region on the other side of the line.



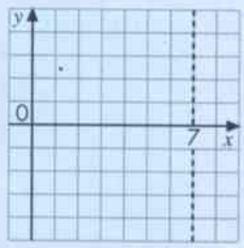
$y \leqslant 3$

b Since the inequality is stated as $>$, the line is *dashed*. Draw the line $x = 7$.

Test the origin (0, 0), which gives $0 > 7$. This is not true, so you want the other side of the line from the origin.

Shade the unwanted region as shown.

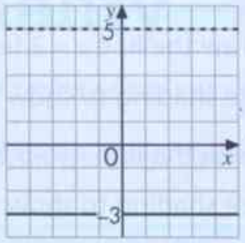
$x > 7$



c Draw the lines $y = -3$ (solid for $\leqslant$) and $y = 5$ (dashed for $<$).

Test a point that is not on either line, say (0, 0). Zero is between –3 and 5, so the required region lies between the lines.

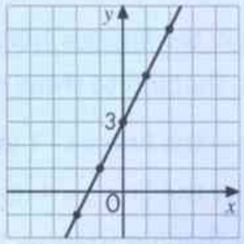
Shade in the unwanted regions, outside these lines.



d Draw the line $y = 2x + 3$. Since the inequality is stated as $\leqslant$, the line is solid.

Test a point that is not on the line, (0, 0). Putting these x- and y-values in the inequality gives $0 \leqslant 2(0) + 3$, which is true. So the region that includes the origin is what you want.

Shade in the unwanted region on the other side of the line.



e Draw the line $2x + 3y = 6$.

The easiest way is to find out where it crosses the axes.

If $x = 0$, $3y = 6 \Rightarrow y = 2$. Crosses y-axis at (0, 2).

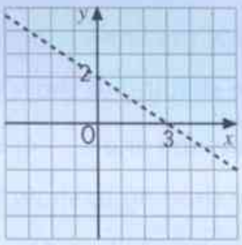
If $y = 0$, $2x = 6 \Rightarrow x = 3$. Crosses x-axis at (3, 0).

Draw the line through these two points.

Since the inequality is stated as $<$, the line is dashed.

Test a point that is not on the line, say (0, 0).
Is it true that $2(0) + 3(0) < 6$? The answer is yes, so the origin is in the region that you want.

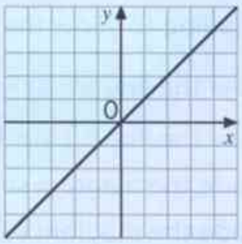
Shade in the unwanted region on the other side of the line.



f Draw the line $y = x$. Since the inequality is stated as $\leqslant$, the line is solid.

This time the origin is on the line, so pick any other point, say (1, 3). Putting $x = 1$ and $y = 3$ in the inequality gives $3 \leqslant 1$. This is not true, so the point (1, 3) is not in the region you want.

Shade in the region that includes (1, 3), the unwanted region.



EXERCISE 20A

EXTENDED

1 a Draw the line $x = 2$ (as a solid line).

b Use shading to show the region defined by $x \leqslant 2$.

2 a Draw the line $y = -3$ (as a dashed line).

b Use shading to show the region defined by $y > -3$.

3 a Draw the line $x = -2$ (as a solid line).

b Draw the line $x = 1$ (as a solid line) on the same grid.

c Use shading to show the region defined by $-2 \leqslant x \leqslant 1$.

4 a Draw the line $y = -1$ (as a dashed line).

b Draw the line $y = 4$ (as a solid line) on the same grid.

c Use shading to show the region defined by $-1 < y \leqslant 4$.

5 a On the same grid, draw the regions defined by these inequalities.

i $-3 \leqslant x \leqslant 6$

ii $-4 < y \leqslant 5$

b Are these points in the region defined by both inequalities?

i (2, 2)

ii (1, 5)

iii (–2, –4)

6 a Draw the line $y = 2x - 1$ (as a dashed line).

b Use shading to show the region defined by $y < 2x - 1$.

7 a Draw the line $3x - 4y = 12$ (as a solid line).

b Use shading to show the region defined by $3x - 4y \leqslant 12$.

8 a Draw the line $y = \frac{1}{2}x + 3$ (as a solid line).

b Use shading to show the region defined by $y \geqslant \frac{1}{2}x + 3$.

9 Use shading to show the region defined by $y < -3$.

10 a Draw the line $y = 3x - 4$ (as a solid line).

b Draw the line $x + y = 10$ (as a solid line) on the same diagram.

c Shade the diagram so that the region defined by $y \geqslant 3x - 4$ is left *unshaded*.

d Shade the diagram so that the region defined by $x + y \leqslant 10$ is left *unshaded*.

e Are these points in the region defined by both inequalities?

i (2, 1)

ii (2, 2)

iii (2, 3)

20.2 More than one inequality

When you have to show a region that satisfies more than one inequality, it is always clearer to *shade* the regions *not required*, so that the *required region* is left *blank*.

Example 2

a On the same grid, show the regions that represent each inequality by shading the unwanted regions.

i $x > 2$

ii $y \geqslant x$

iii $x + y < 8$

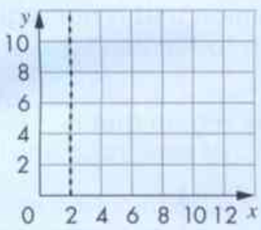
b Are these points in the region that satisfies all three inequalities?

i (3, 4)

ii (2, 6)

iii (3, 3)

a i



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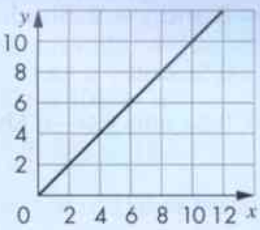

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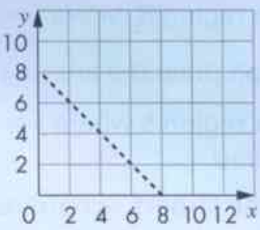
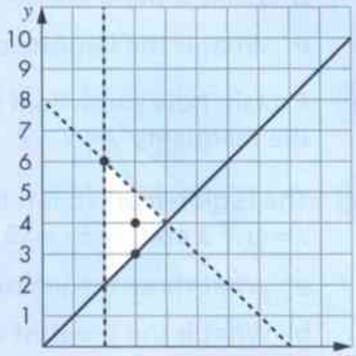


i The region $x > 2$ is shown unshaded in diagram **i**.
The boundary line is $x = 2$ (dashed).

ii The region $y \geqslant x$ is shown unshaded in diagram **ii**.
The boundary line is $y = x$ (solid).

iii The region $x + y < 8$ is shown unshaded in diagram **iii**.

The boundary line is $x + y = 8$ (dashed). The regions have first been drawn separately so that each may be clearly seen. The diagram on the right shows all three regions on the same grid. The white triangular area defines the region that satisfies all three inequalities.



b i The point (3, 4) is clearly within the region that satisfies all three inequalities.

ii The point (2, 6) is on the boundary lines $x = 2$ and $x + y = 8$. As these are dashed lines, they are not included in the region defined by all three inequalities. So, the point (2, 6) is not in this region.

iii The point (3, 3) is on the boundary line $y = x$. As this is a solid line, it is included in the region defined by all three inequalities. So, the point (3, 3) is included in this region.

EXERCISE 20B

EXTENDED

1 a Draw the line $y = x$ (as a solid line).

b Draw the line $2x + 5y = 10$ (as a solid line) on the same diagram.

c Draw the line $2x + y = 6$ (as a dashed line) on the same diagram.

Advice and Tips

Find the points where the line crosses each axis.

d Shade the diagram so that the region defined by $y \geqslant x$ is left *unshaded*.

e Shade the diagram so that the region defined by $2x + 5y \geqslant 10$ is left *unshaded*.

f Shade the diagram so that the region defined by $2x + y < 6$ is left *unshaded*.

g Are these points in the region defined by these inequalities?

i (1, 1) ii (2, 2) iii (1, 3)

2 a On the same grid, draw the regions defined by the following inequalities. (Shade the diagram so that the overlapping region is left blank.)

i $y > x - 3$ ii $3y + 4x \leqslant 24$ iii $x \geqslant 2$

b Are these points in the region defined by all three inequalities?

i (1, 1) ii (2, 2) iii (3, 3) iv (4, 4)

3 a On a graph draw the lines $y = x$, $x + y = 8$ and $y = 2$.

b Label the region R where $y \leqslant x$, $x + y \leqslant 8$ and $y \geqslant 2$. Shade the region that is *not* required.

4 a On a graph draw the lines $y = x - 4$, $y = 0.5x$ and $y = -x$.

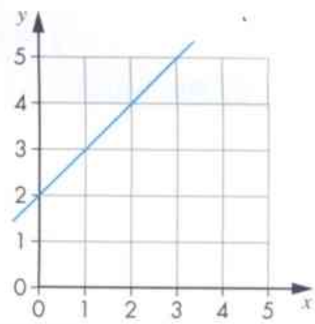
b Show the region S where $y \geqslant x - 4$, $y \leqslant 0.5x$ and $y \geqslant -x$. Shade the region that is *not* required.

c What is the largest y-coordinate of a point in S?

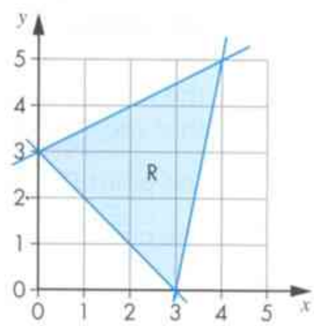
d What is the smallest y-coordinate of a point in S?

e What is the largest value of $x + y$ for a point in S?

5 Explain how you would find which side of the line represents the inequality $y < x + 2$.



6 The region marked R is bounded by the lines $x + y = 3$, $y = \frac{1}{2}x + 3$ and $y = 5x - 15$.



a What three inequalities are satisfied in region R?

b What is the greatest value of $x + y$ in region R?

c What is the greatest value of $x - y$ in region R?

20.3 Linear programming

E

There are practical situations that give rise to inequalities that can be shown on a graph. Solving problems in this way is called **linear programming**.

Example 3

A boat trip costs \$20 for adults and \$10 for children.

Suppose there are x adults and y children on the trip.

a There must not be more than 10 people on the trip.
Explain why $x + y \leqslant 10$.

b The money taken for tickets must be at least \$120.
Explain why $2x + y \geqslant 12$.

c Show the region satisfied by both these inequalities.

d What is the smallest number of adults that should be carried?

a The total number of passengers is 10.

The total must be 10 or less so $x + y \leqslant 10$.

b The total number of dollars taken is $20x + 10y$.

This must be at least 120 so $20x + 10y \geqslant 120$.

Divide both sides by 10 to get $2x + y \geqslant 12$.

c Negative numbers on the axes are not needed.

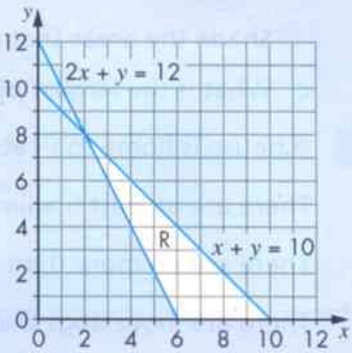
First draw the lines with equations $x + y = 10$ and $2x + y = 12$.

The first crosses the axes at (10, 0) and (0, 10).

The second crosses the axes at (6, 0) and (0, 12).

Shade the regions that are *not* required.

The solution must be in the unshaded region R.



d The smallest x-coordinate in this region is at (2, 8).

x must be at least 2 so there must be at least 2 adults.

EXERCISE 20C

EXTENDED

1 A man buys x cartons of milk and y bottles of water.

- **a** Explain what $x \geqslant 3$ means.
- **b** Write an inequality to show that he buys at least 2 bottles of water.
- **c** He cannot carry more than a total of 8 bottles and cartons. Write this as an inequality.
- **d** Draw a graph to show possible values of x and y.

2 There are x cars and y vans in a car park.

- **a** There are no more than 10 vehicles (cars and vans) in the car park.

  Write an inequality to show this.
- **b** Cars pay \$2 and vans pay \$5 to park. At least \$30 was paid in total.

  Write an inequality to show this.
- **c** Show the two inequalities on a grid.
- **d** What is the largest possible number of cars?
- **e** What is the smallest possible number of vans?

3 x girls and y boys are at a children's party.

There are more girls than boys.

The number of girls is fewer than twice the number of boys.

There are fewer than 12 children at the party.

- **a** Write down three inequalities to represent each of the statements above.
- **b** Show the region indicated by these inequalities.

  Shade the areas that are not included.
- **c** What is the largest possible number of girls at the party?

4 A house is home to x cats and y dogs.

There are at least 6 animals.

There are no more than 12 animals.

There are more dogs than cats.

There are at least two cats.

- **a** Write down four inequalities from these sentences.
- **b** Show the region that gives possible values of x and y by shading the area that is not required.
- **c** What is the smallest possible number of dogs?
- **d** What is the largest possible number of cats?

EXTENDED

5 At a concert, x people bought seats near the front for \$15 and y people bought seats near the back for \$10.

a Ticket sales were at least \$600. Show that $3x + 2y \geqslant 120$.

b No more than 50 tickets were sold. Write an inequality for this.

c Show the region which represents these inequalities on a grid.

d What is the largest possible number of back seats sold?

6 A company has a fleet of coaches in two sizes. There are x large coaches and y small coaches.

a Write an inequality for each of these statements.

i The total number of coaches must not exceed 15.

ii There must be at least 5 coaches in total.

iii The number of small coaches must not be less than half the number of large coaches.

b Explain in words the meaning of $y \leqslant x + 2$.

c Label the region R to show the four inequalities in parts **a** and **b**. Shade the area which is not required.

d If there are 6 large coaches, how many small coaches could there be? Give all the possible values.

7 In a football league a team score 3 points for a win, 1 point for a draw and none for a loss.

a If a team wins x matches and draws y matches, write down an expression for the total number of points gained.

b Wayne knows these facts about his team's performance so far this season.

- They have gained at most 18 points.
- They have more wins than draws.
- They have at least two draws.

Write an inequality, using x and y, for each of these three statements.

c Show on a graph the region where the three inequalities are satisfied.

d List all the possible numbers of wins and draws for Wayne's team.

Check your progress

Extended

- I can represent inequalities graphically
- I can use graphical representations to solve simple linear programming problems

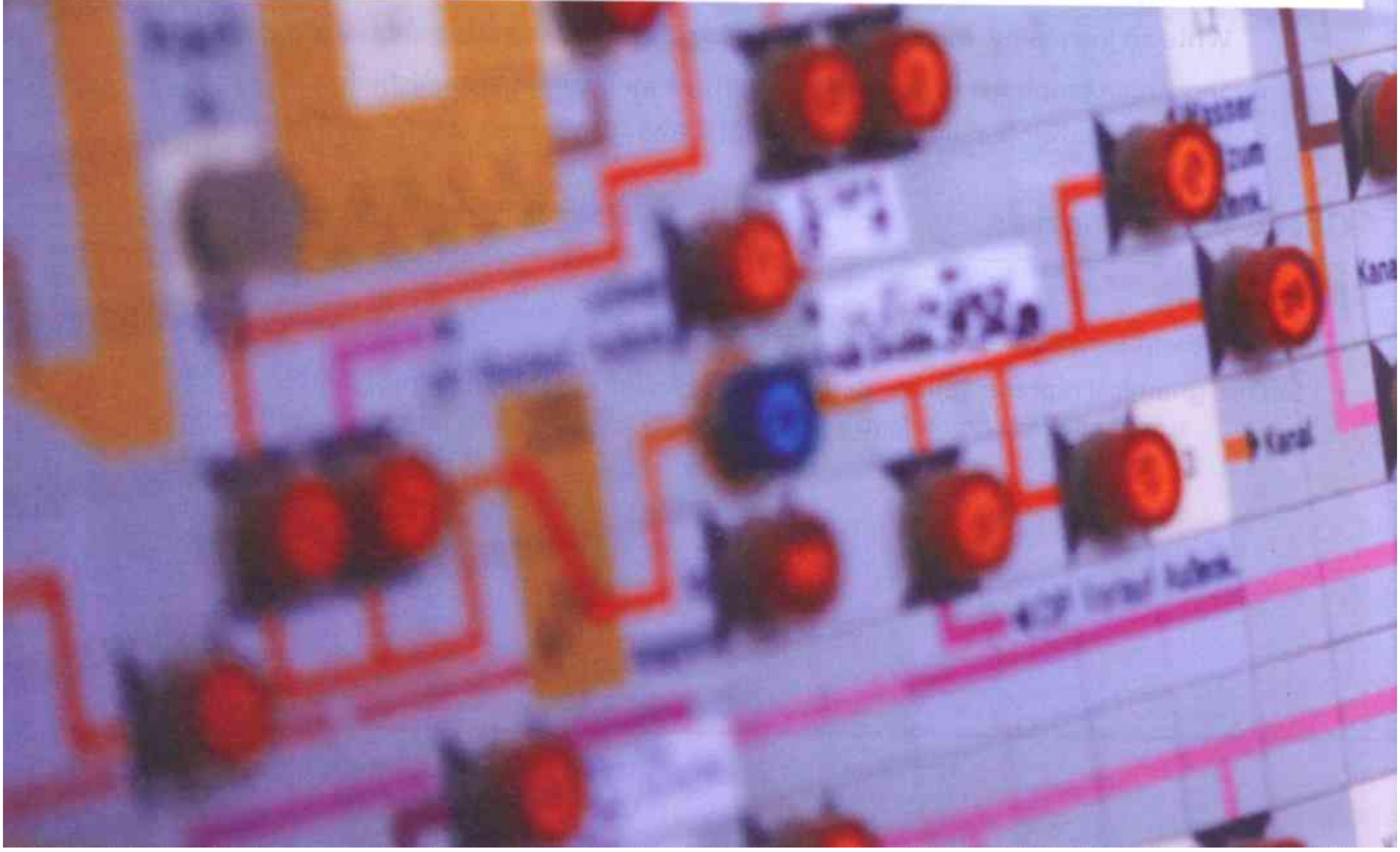
Chapter 21
Functions

Topics	Level	Key words
1 Function notation	EXTENDED	function
2 Inverse functions	EXTENDED	inverse
3 Composite functions	EXTENDED	composite
4 More about composite functions	EXTENDED	

In this chapter you will learn how to:

EXTENDED

- Use function notation, e.g. $f(x) = 3x - 5$, $f: x \to 3x - 5$ to describe simple functions. (E2.9)
- Find inverse functions $f^{-1}(x)$. (E2.9)
- Form composite functions as defined by $gf(x) = g(f(x))$. (E2.9)



Why this chapter matters

Notation is important in mathematics. Try writing an equation or formula in words instead of symbols and you will see why notation makes things easier to understand.

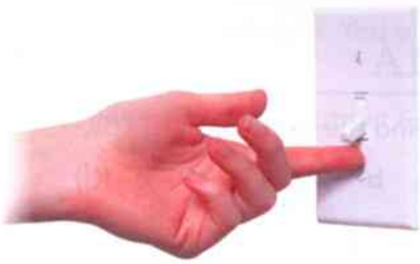
If you drop a coin, how long does it take to reach the ground? That depends on the height you drop it from. We say that the time taken is a **function** of the height.

By dropping the coin from different heights and measuring the time it takes to fall, it would be possible to find a formula for the time in terms of the height.

Here are some other examples where one variable is a **function** of another.

- The cost of posting a parcel is a function of its mass.
- The time taken for a journey is a function of the distance travelled.
- The stopping distance of a car is a function of its speed.
- The cost of a second hand car is a function of its age.
- The time taken to download a computer file is a function of the size of the file.

The idea of an inverse occurs frequently in mathematics. It is not a difficult idea and it is a useful one. Putting a hat on and taking it off are inverse operations. Switching a light on and switching it off are also inverse operations.



Here are some examples of inverse operations in mathematics.

- Add 3 and subtract 3.
- Multiply by 5 and divide by 5.
- Rotate 90° clockwise and rotate 90° anticlockwise.
- Square a number and find the square root of the square number.

It is not always possible to find an inverse. Sadly the inverse of breaking a glass does not exist.

21.1 Function notation

E

You are familiar with equations written in x and y, such as $y = 3x - 4$ or $y = 2x^2 + 5x - 3$.

These equations are showing that y is a function of x. This means that the value of y depends on the value of x so that y changes when x changes.

Sometimes it is useful to use a different notation to show this. You could write the first equation above as $f(x) = 3x - 4$ and refer to it as 'function f'.

It is then easy to show the result of using different values for x. For example:

- 'the value of $f(x)$ when x is 5' can be written as f(5).

  So $f(5) = 3 \times 5 - 4 = 11$

- f(1) means 'the value of $f(x)$ when x is 1'

  So $f(1) = 3 \times 1 - 4 = -1$

  and $f(-1) = -7$

If there are different functions in the same problem they can be represented by different letters, for example:

$g(x) = 2x^2 + 5x - 3$ or 'function g'.

Sometimes instead of $f(x) = 3x - 4$ you may see $f: x \to 3x - 4$. These two forms mean exactly the same thing.

EXERCISE 21A

EXTENDED

1 $f(x) = 2x + 6$. Find the value of:

a $f(3)$ **b** $f(10)$ **c** $f(\frac{1}{2})$ **d** $f(-4)$ **e** $f(-1.5)$.

2 $g(x) = \frac{x^2 + 1}{2}$. Find the value of:

a $g(0)$ **b** $g(3)$ **c** $g(10)$ **d** $g(-2)$ **e** $g(-\frac{1}{2})$.

3 $f: x \to x^3 - 2x + 1$. Find the value of:

a $f(2)$ **b** $f(-2)$ **c** $f(100)$ **d** $f(0)$ **e** $f(\frac{1}{2})$.

4 $g: x \to 2^x$. Find the value of:

a $g(2)$ **b** $g(5)$ **c** $g(0)$ **d** $g(-1)$ **e** $g(-3)$.

5 $h(x) = \frac{x + 1}{x - 1}$. Find the value of:

a $h(2)$ **b** $h(3)$ **c** $h(-1)$ **d** $h(0)$ **e** $h(1\frac{1}{2})$.

6 $f: x \to 2x + 5$

a If $f(a) = 20$, what is the value of a?

b If $f(b) = 0$, what is the value of b?

c If $f(c) = -5$, what is the value of c?

7 $g(x) = \sqrt{x+3}$

a Find the value of g(33).

b If $g(a) = 10$, find the value of a.

c If $g(b) = 2.5$, find the value of b.

8 $f(x) = 2x - 8$ and $g(x) = 10 - x$.

a What is the value of x for which $f(x) = g(x)$?

b Sketch the graphs of $y = f(x)$ and $y = g(x)$. At what point do they cross?

9 $h: x \to \frac{12}{x} + 1$ $\quad$ $k: x \to 2^x - 1$

a Find the value of h(6).

b Find the value of k(–1).

c Solve the equation $h(x) = k(3)$.

d Solve the equation $k(x) = h(-12)$.

21.2 Inverse functions

E

Suppose $f(x) = 2x + 6$.

Then $f(1) = 8$, $f(3) = 12$ and $f(-4) = -2$.

The **inverse** of f is the function that has the opposite effect and 'undoes' f. You write the inverse of f as f^{-1}.

Since f above means 'multiply by 2 and then add 6', the inverse will be 'subtract 6 and then divide by 2':

$$f^{-1}(x) = \frac{x-6}{2}$$

So $f^{-1}(8) = 1$, $f^{-1}(12) = 3$ and $f^{-1}(-2) = -4$.

You can find the inverse by following these steps.

Step 1: Write $y = f(x)$

$$y = 2x + 6$$

Step 2: Rearrange to make x the subject.

$$y - 6 = 2x$$

$$\frac{y-6}{2} = x$$

$$x = \frac{y-6}{2}$$

Step 3: Replace y by x in the result.

$$f^{-1}(x) = \frac{x-6}{2}$$

EXERCISE 21B

EXTENDED

1 Find $f^{-1}(x)$ for each of these functions.

a $f(x) = x + 7$

b $f(x) = 8x$

c $f(x) = \frac{x}{5}$

d $f(x) = x - 3$

2 $f: x \rightarrow \frac{x}{2} + 6$. Find the value of:

a $f(4)$

b $f^{-1}(8)$

c $f(-2)$

d $f^{-1}(5)$.

3 Find $f^{-1}(x)$ for each of these functions.

a $f(x) = \frac{x}{3} - 2$

b $f(x) = 4(x - 5)$

c $f(x) = \frac{x + 4}{5}$

d $f(x) = \frac{(3x - 6)}{5}$

e $f(x) = 3(\frac{x}{2} + 4)$

f $f(x) = 4x^3$

4 $g(x) = \frac{2x + 5}{3}$. Find the value of:

a $g^{-1}(3)$

b $g^{-1}(2)$

c $g^{-1}(0)$.

5 $f(x) = 10 - x$

a Find an expression for $f^{-1}(x)$.

b What do you notice about $f(x)$ and $f^{-1}(x)$?

6 Find $f^{-1}(x)$ for each of these functions.

a $f(x) = \frac{8}{x}$

b $f(x) = \frac{20}{x} - 1$

c $f(x) = \frac{2}{x + 1}$

7 $f(x) = 2x - 4$

a Find $f^{-1}(x)$.

b On the same axes draw graphs of $y = f(x)$ and $y = f^{-1}(x)$.

c Where do the lines cross?

8 $f(x) = \frac{x + 5}{2}$

Solve the equation $f(x) = f^{-1}(x)$.

9 $f^{-1}(x) = 3x - 2$

Find an expression for $f(x)$.

21.3 Composite functions

E

Consider a function given by $f(x) = 2x$.

So f means 'double it'.

f(2) = 4

f(3) = 6

f(5) = 10

and f(–3) = –6

Now suppose you have another function, g, and $g(x) = x - 3$.

So g means 'subtract 3'.

g(4) = 1

g(6) = 3

g(10) = 7

and g(–6) = –9

Now put those side by side to make a **composite** function.

f(2) = 4	and	g(4) = 1
f(3) = 6	and	g(6) = 3
f(5) = 10	and	g(10) = 7
f(–3) = –6	and	g(–6) = –9

In words they mean:

Start with 2, double it, subtract 3, the answer is 1.

Start with 3, double it, subtract 3, the answer is 3.

Start with 5, double it, subtract 3, the answer is 7.

Start with –3, double it, subtract 3, the answer is –9.

You can write that in symbols as:

gf(2) = 1

gf(3) = 3

gf(5) = 7

gf(–3) = –9

Advice and Tips

gf means 'first f, then g'.

Example 1

h(x) = x^2 and k(x) = x + 4. Find:

a kh(3)

b kh(–2)

c hk(5)

a h means 'square it' and k means 'add 4'.

kh(3) means 'start with 3, square it, then add 4'.

So kh(3) = 3^2 + 4 = 9 + 4 = 13

b kh(–2) = $(-2)^2$ + 4 = 4 + 4 = 8

c hk(5) is the other way round.

It means 'start with 5, add 4, then square it'.

hk(5) = $(5 + 4)^2$ = 9^2 = 81

Look again at the functions f and g you considered earlier.

f(x) = 2x and g(x) = x –3

gf(x) means 'first double x, then subtract 3'. So you can write:

gf(x) = 2x – 3

What about fg(x)?

That means 'first subtract 3, then double it'. So you can write:

fg(x) = 2(x – 3)

Looking at example 1, h(x) = x^2 and k(x) = x + 4.

Then kh(x) = x^2 + 4

and hk(x) = $(x + 4)^2$

EXERCISE 21C

EXTENDED

1 s(x) = x + 4 and t(x) = $\frac{x}{2}$

a Find the value of s(2) and ts(2).

b Find the value of s(3) and ts(3).

c Find the value of s(6) and ts(6).

d Find an expression for ts(x).

e Find the value of t(2) and st(2).

f Find the value of t(3) and st(3).

g Find the value of t(–10) and st(–10).

h Find an expression for st(x).

2 c(x) = x^3 and d(x) = 2x.

a Find the value of d(3) and cd(3).

b Find the value of d(5) and cd(5).

c Find an expression for cd(x).

d Find the value of c(4) and dc(4).

e Find an expression for dc(x).

3 $r(x) = \sqrt{x}$ and $a(x) = 2x + 1$

a Find the value of a(0), a(4) and a(12). **b** Find the value of ra(0), ra(4) and ra(12).

c Find an expression for $ra(x)$.

4 $m(x) = 3x$

a Find the value of m(2) and mm(2). **b** Find the value of m(4) and mm(4).

c Find an expression for $mm(x)$.

5 $f(x) = 3x$ and $g(x) = x - 6$

a Find an expression for $fg(x)$. **b** Find an expression for $gf(x)$.

6 $a(x) = x + 4$ and $b(x) = x - 7$

Show that $ab(x)$ and $ba(x)$ are identical.

21.4 More about composite functions

E

Suppose $f(x) = x^2 + 2$ and $g(x) = 2x - 3$.

Is it possible to find an expression for $gf(x)$?

Start with a value of x, say $x = 3$.

$$f(3) = 3^2 + 2 = 11$$

Now apply g to that answer.

$$g(11) = 2 \times 11 - 3 = 19$$

So $gf(3) = 19$

$gf(x)$ means 'start with x, apply f, and then apply g to the answer.'

If you start with x and 'apply f' you get $x^2 + 2$.

Now take that answer and 'apply g', double it and subtract 3.

$$gf(x) = 2(x^2 + 2) - 3$$

Now simplify that.

$$gf(x) = 2(x^2 + 2) - 3$$
$$= 2x^2 + 4 - 3$$
$$= 2x^2 + 1$$

So $gf(x) = 2x^2 + 1$

Check this works when x is 3.

$$gf(3) = 2 \times 3^2 + 1$$
$$= 18 + 1$$
$$= 19 \text{ as before.}$$

EXERCISE 21D

EXTENDED

1 $f(x) = \frac{x-3}{2}$ and $g(x) = 3x + 1$

Find the value of:

a fg(3)

b gf(3)

c fg(6)

d gf(6).

2 $f(x) = x^2 - x$ and $g(x) = \frac{x}{2} + 3$

Find the value of:

a fg(4)

b gf(4)

c fg(1)

d gf(1).

3 $f(x) = 2^x$ and $g(x) = 2x - 1$

Find the value of:

a gf(2)

b fg(2)

c ff(3)

d gg(6).

4 $f(x) = 3x + 1$ and $g(x) = 2x - 2$

a Find an expression for $gf(x)$. Write your answer as simply as possible.

b Find an expression for $fg(x)$. Write your answer as simply as possible.

5 In each case find an expression for $fg(x)$. Write your answer as simply as possible.

a $f(x) = x^2$ and $g(x) = 3x + 4$

b $f(x) = 2x + 3$ and $g(x) = 3x - 4$

c $f(x) = \frac{x}{2} + 4$ and $g(x) = 4x - 2$

d $f(x) = 12 - x$ and $g(x) = 2x + 8$

6 $h(x) = 10 - x$ and $k(x) = 20 - x$

Find an expression for:

a $hk(x)$

b $kh(x)$

c $kk(x)$.

7 $h(x) = x^2$ and $k(x) = \frac{12}{x}$

Find an expression for:

a $hh(x)$

b $hk(x)$

c $kh(x)$

d $kk(x)$.

8 $m(x) = x^2 + 2x$ and $n(x) = 2x - 1$

a Find the value of mm(2).

b Show that $nn(x) = 4x - 3$.

c Show that $mn(x) = 4x^2 - 1$

9 $f(x) = \frac{1}{x-4}$ and $g(x) = 3x + 1$.

a Find an expression for $fg(x)$.

b Find an expression for $gf(x)$, writing your answer as a single fraction.

10 $h(x) = \frac{x+4}{2}$ and $k(x) = 3x - 5$

Find:

a $h^{-1}k(x)$

b $hh(x)$.

11 Suppose $f(x) = 0.5(x + 9)$.

a Show that $f(1) = 5$.

b Find the value of f(5).

c Find $f(b)$ where b is the answer to part **b**.

d Continue in this way, using the last answer as the next value of x, to find the next six values.

e What is happening to the answers?

12 What happens in question 11 if you start with f(25) instead of f(1)?

Check your progress

Extended

- I can use function notation to describe simple functions
- I can find inverse functions
- I can find composite functions

Chapter 22

Differentiation

Topics	Level	Key words
1 The gradient of a curve	EXTENDED	gradient
2 More complex curves	EXTENDED	derivative, differentiate, differentiation
3 Turning points	EXTENDED	turning point, maximum, minimum

In this chapter you will learn how to:

EXTENDED

- Differentiate integer powers of x and simple sums of these. (E2.13)
- Understand the idea of a derived function. (E2.13)
- Use the derivatives of functions of the form ax^n, and simple sums of not more than three of these. (E2.13)
- Determine gradients, and turning points (maxima and minima) by differentiation and relate these to graphs. (E2.13)
- Distinguish between maxima and minima by considering the general shape of the graph. (E2.13)

Why this chapter matters

When you look at a graph, it shows you how changes in one variable lead to changes in another. Differentiation helps us to find the rate at which this change happens.

When you look at a straight line graph you can easily calculate the gradient which tells you how quickly one variable changes compared to another. But when the line on a graph is curved, the gradient is changing all the time. Using differentiation you can calculate the gradient and so work out how quickly change is happening at any point.

For example:

- When you walk up a hill your height compared to ground level changes with respect to your position on the slope
- When you drive a car your position and your velocity vary with respect to time
- The brightness of a light bulb varies depending on the electric current flowing through it
- Hot drinks cool down and ice melts at different rates as time passes

We can use differentiation to understand how changes like this happen.

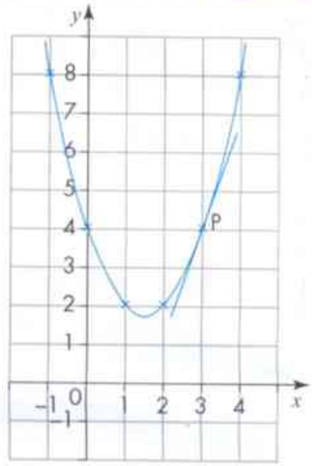
A form of differentiation was used in Ancient Greece and China. In the seventeenth century CE a German mathematician called Leibniz and an English scientist, Isaac Newton, started to develop the form of differentiation we use today.

Differentiation is now an essential tool in almost all areas of science, including engineering, physics, chemistry, biology, economics and computer science.

22.1 The gradient of a curve

E

This is a graph of $y = x^2 - 3x + 4$.



What is the **gradient** at the point P(3, 4)?

In an earlier chapter you learnt how to find the gradient by drawing a tangent to the curve at P. It is hard to do this accurately and we need to draw the graph first.

We will now look at another method.

The point A(4, 8) is on the curve.

The gradient of the straight line PA is

$$\frac{\text{difference in } y \text{ - coordinates}}{\text{difference in } x \text{ - coordinates}} = \frac{8-4}{4-3} = 4$$

The line PA is steeper than the tangent so the gradient of the curve at P is less than 4.

Suppose A is a point on the curve closer to P.

If A is (3.5, 5.75) Then the gradient of PA is $\frac{5.75-4}{3.5-3} = 3.5$

If A is (3.1, 4.31) Then the gradient of PA is $\frac{4.31-4}{3.1-3} = 3.1$

If A is (3.01, 4.0301) Then the gradient of PA is $\frac{4.0301-4}{3.01-3} = 3.01$

As A gets closer to P, the gradient of PA approaches 3.

It is reasonable to assume that the gradient of the tangent at P(3, 4) is 3

This method could be used at any point on the curve and it will show the following:

The gradient at any point on the curve $y = x^2 - 3x + 4$ is $2x - 3$.

For example, at (3, 4) the gradient is $2 \times 3 - 3 = 3$

at (4, 8) the gradient is $2 \times 4 - 3 = 5$

at (1, 2) the gradient is $2 \times 1 - 3 = -1$

Looking at the graph should convince you that these seem to be reasonable values.

The notation we use to represent the gradient of a curve is $\frac{dy}{dx}$ (read it as "dee y by dee x"). So the tangent to the curve on the graph shows that:

If $y = x^2 - 3x + 4$

$\frac{dy}{dx} = 2x - 3$

This is the case for all equations of this form. So the general result is:

If $y = ax^2 + bx + c$

then $\frac{dy}{dx} = 2ax + b$

We can use this to calculate the gradient at any point on a quadratic curve. We no longer need to draw the graph and this method can be used for any point on the curve.

Example 1

A curve has the equation $y = 0.5x^2 + 4x - 3$

a Find the gradient at (0, −3) and at (2, 7).

b Find the coordinates of the point where the gradient is 0.

a Using the general result above, if $y = 0.5x^2 + 4x - 3$,

then $\frac{dy}{dx} = 2 \times 0.5x + 4$

$\Rightarrow \frac{dy}{dx} = x + 4$

If $x = 0$, $\frac{dy}{dx} = 0 + 4 = 4$. The gradient at (0, −3) is 4

If $x = 2$, $\frac{dy}{dx} = 2 + 4 = 6$. The gradient at (2, 7) is 6

b If the gradient is 0, then $\frac{dy}{dx} = 0$

$\Rightarrow x + 4 = 0$

$\Rightarrow x = -4$

If $x = -4$, $y = 0.5 \times (-4)^2 + 4 \times (-4) - 3 = -11$

$\Rightarrow$ The gradient is 0 at (−4, −11)

EXERCISE 22A

1 A curve has the equation $y = x^2 - 2x$.

a Copy and complete this table of values:

x	−2	−1	0	1	2	3	4
$x^2 - 2x$	8	3				3	8

b Sketch the graph of $y = x^2 - 2x$.

c Find $\frac{dy}{dx}$.

d Find the gradient of the curve at (3, 3).

e Find the gradient of the curve at (4, 8).

f Find the gradient at two more points on the curve.

g At what point on the graph is the gradient 0?

h By looking at your graph, check that your answers to parts **d**, **e**, **f** and **g** seem sensible.

2 $y = x^2 - 6x + 15$

a Find $\frac{dy}{dx}$.

b Find the gradient at (0, 15).

c Find the gradient at (5, 10).

d Find the coordinates of the point where the gradient is 2.

EXTENDED

EXTENDED

3 $y = 2x^2 - 10$

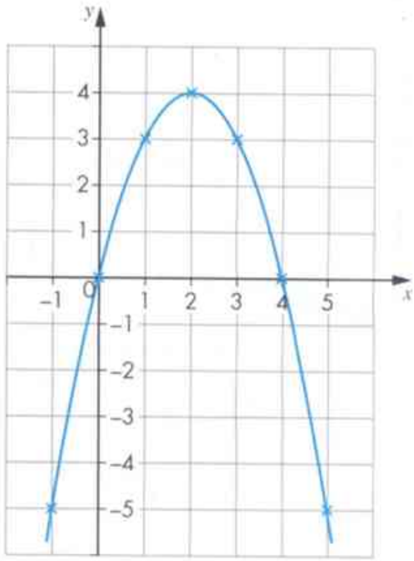
a Find $\frac{dy}{dx}$.

b Find the gradient at (2, −2).

c Find the gradient at (−1, −9).

d Find the point where the gradient is 12.

4 This is the graph of $y = 4x - x^2$



a Find $\frac{dy}{dx}$.

b Find the gradient at each point where the curve crosses the x-axis.

c Where is the gradient equal to 2?

d Where is the gradient equal to 1?

5 Find $\frac{dy}{dx}$ for each of the following:

a $y = x^2 + x + 1$

b $y = x^2 - 7x + 3$

c $y = 4x^2 - x + 6$

d $y = 0.3x^2 - 1.5x + 7.2$

e $y = 6 - 2x + x^2$

f $y = 10 + 3x - x^2$

g $y = 2x + 5$

h $y = 4$

6 If $y = (x + 4)(x - 2)$ what is $\frac{dy}{dx}$?

7 Find $\frac{dy}{dx}$ for each of the following:

a $y = 2x(x + 1)$

b $y = (x + 2)(x + 5)$

c $y = (x + 3)(x - 3)$

Advice and Tips

First multiply out the brackets.

8 A curve has the equation $y = x^2 + 2x - 5$

a Where does the curve cross the y-axis?

b What is the gradient of the curve at that point?

22.2 More complex curves

E

You have seen that if $y = x^2$ then $\frac{dy}{dx} = 2x$.

This table shows the value of $\frac{dy}{dx}$ for some other curves.

y	$\frac{dy}{dx}$	
1	0	→ The lines $y = 1$ has gradient 0
x	1	→ The line $y = x$ has gradient 1
x^2	$2x$	

x^3	$3x^2$
x^4	$4x^3$

There is a general pattern here:

If $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$

We call $\frac{dy}{dx}$ the derivative of y with respect to x.

If a is a constant and $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$

For example, if $y = 5x^2$ then $\frac{dy}{dx} = 5 \times 2x = 10x$. If $y = 4x^3$ then $\frac{dy}{dx} = 12x^2$.

If $y = 6$ then $\frac{dy}{dx} = 0$. So the line with equation $y = 6$ is horizontal and has gradient 0.

Example 2

What is the gradient of the curve with equation $y = x^3 - 3x^2 + 4x + 7$ at the point (2, −5)?

$\frac{dy}{dx} = 3x^2 - 6x + 4$

If $x = 2$, $\frac{dy}{dx} = 3 \times 2^2 - 6 \times 2 + 4 = 4$

The gradient at (2, −5) is 4.

The process of finding $\frac{dy}{dx}$ is called **differentiation**.

In example 2 we **differentiated** each term in turn:

Differentiate x^3 to get $3x^2$

Differentiate $-3x^2$ to get $-6x$

Differentiate $4x$ to get 4

Differentiate 7 to get 0

$\frac{dy}{dx} = 3x^2 - 6x + 4$

Note the minor change to the wording so that it now says "We call $\frac{dy}{dx}$ the derivative of y with respect to x"

EXERCISE 22B

1 The equation of a curve is $y = 2x^3$.

a Find $\frac{dy}{dx}$.

b Find the gradient of the curve at (1, 2) and (2, 16).

2 The equation of a curve is $y = x^3 - 6x^2 + 8x$.

a Find $\frac{dy}{dx}$.

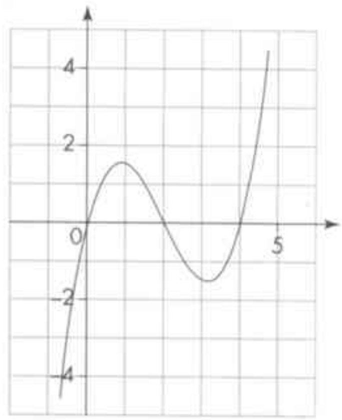
b Show that (0, 0), (2, 0) and (4, 0) are all on this curve.

c Find the rate of change of y with respect to x at each point in part **b**.

EXTENDED

EXTENDED

3 This is a graph of $y = 0.5x^3 - 3x^2 + 4x$



a Find $\frac{dy}{dx}$

b Find the gradient at each point where the graph crosses the x-axis.

4 Find $\frac{dy}{dx}$ for each of the following:

a $y = 2x^4$

b $y = 2x^3 + 5x^2 - 8$

c $y = 5x^3 - 2x + 4$

d $y = 5 - x - \frac{2}{3}x^3$

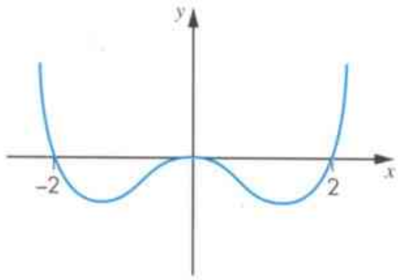
e $y = 3x^3 + 5x - 7$

f $y = 10 - x^3$

g $y = x(x^3 - 1)$

h $y = 2x^3(x + 3)$

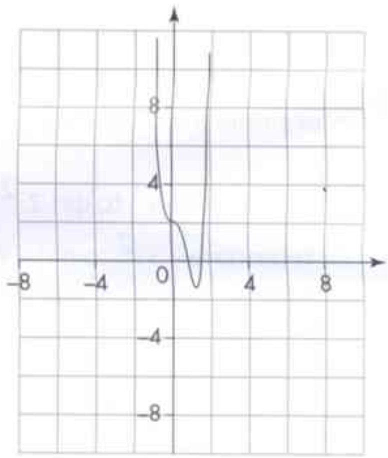
5 This is a sketch of the curve $y = x^4 - 4x^2$.



Not to scale

Find the gradient at the points where the curve meets the x-axis.

6 This is a graph of $y = x^4 - 2x^3 + 1$



a Show that the gradient at (0, 1) is 0

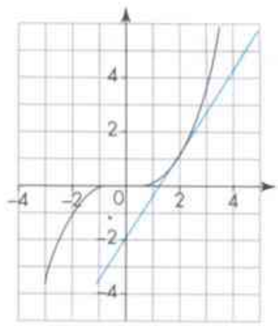
b Find the gradient at (–1, 4)

c Find the gradient at (2, 1)

7 A curve has the equation $y = \frac{1}{3}x^3 - 5x + 4$.

Show that there are two points on the curve where the rate of change of y with respect to x is 4. Find the coordinates of the two points.

8 This is a graph of the curve $y = \frac{1}{8}x^3$ and the tangent at (2, 1)



Find the equation of the tangent at (2, 1)

9 The equation of a curve is $y = x^3 - 2ax$ where a is a constant.

When $x = 2$ the gradient of the curve is −12

a Find the value of a

b Find the gradient when $x = 4$

22.3 Turning points

E

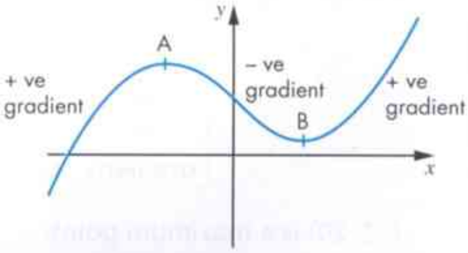
A point where the gradient is zero is called a **turning point.**

A and B are turning points.

At any turning point $\frac{dy}{dx} = 0$

A is called a **maximum** point because it is higher than the points near it.

B is called a **minimum** point.



Example 3

Find the turning points of $y = x^3 - 12x + 4$ and state whether each is a maximum or a minimum point.

If $y = x^3 - 12x + 4$:

$\frac{dy}{dx} = 3x^2 - 12$

At a turning point $\frac{dy}{dx} = 0$:

$\Rightarrow \quad 3x^2 - 12 = 0$

$\Rightarrow \quad 3x^2 = 12$

$\Rightarrow \quad x^2 = 4$

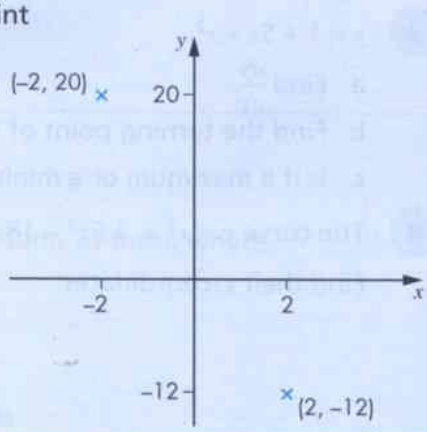
$\Rightarrow \quad x = 2$ or -2

If $x = 2$, $y = 8 - 24 + 4 = -12$ $\quad\Rightarrow\quad$ (2, −12) is a turning point

If $x = -2$, $y = -8 + 24 + 4 = 20$ $\quad\Rightarrow\quad$ (−2, 20) is a turning point

A rough sketch makes it look likely that (−2, 20) is a maximum point and (2, −12) is a minimum point.

If you are not sure, check the gradient on each side of the point.



For (2, −12):

x	1.9	2	2.1
$\frac{dy}{dx}$	−1.17	0	1.23
gradient	negative	0	positive

This is $3 \times 2.1^2 - 12$

(2, −12) is a minimum point.

Because the gradient changes (from negative to zero to positive) as we move from left to right, (2, −12) must be a minimum point.

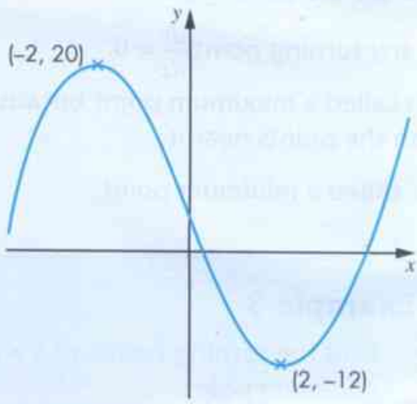
For (−2, 20):

x	−2.1	−2	−1.9
$\frac{dy}{dx}$	1.23	0	−1.17
gradient	positive	0	negative

(−2, 20) is a maximum point.

Because the gradient changes (from positive to zero to negative) as we move from left to right, (−2, 20) must be a maximum point.

Here is a sketch of the curve.



EXERCISE 22C

EXTENDED

1 $y = x^2 - 4x + 3$

a Find $\frac{dy}{dx}$.

b Show that the curve has one turning point and find its coordinates.

c State whether it is a maximum or minimum point.

2 a Find the turning point of the curve $y = x^2 + 6x - 3$.

b Is it a maximum or a minimum point?

3 $y = 1 + 5x - x^2$

a Find $\frac{dy}{dx}$.

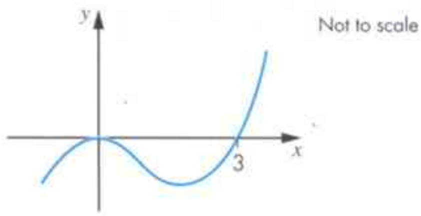
b Find the turning point of the curve.

c Is it a maximum or a minimum point?

4 The curve $y = x^3 + 1.5x^2 - 18x$ has two turning points.

Find their x-coordinates.

5 This is a sketch of $y = x^3 - 3x^2$



a Find $\frac{dy}{dx}$.

b Solve the equation $\frac{dy}{dx} = 0$.

c Find the coordinates of the two turning points shown on the graph.

6 $y = x^2 - 3x - 10$

a Show that the graph crosses the x-axis at (–2, 0) and (5, 0).

b Find $\frac{dy}{dx}$.

c Find the turning point of the curve. Is it a maximum or a minimum?

d The curve has a line of symmetry. What is the equation of this line?

7 The curve $y = 18x^2 - x^4$ has three turning points.

a Find their coordinates.

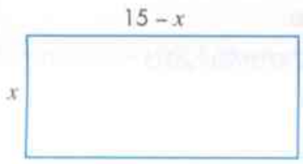
b Sketch the graph.

8 The equation of a curve is $y = 2x^3 - 6x + 4$.

a Find $\frac{dy}{dx}$.

b Find the turning points of the curve and state whether each one is a maximum or a minimum point.

9 A rectangle has a perimeter of 30 cm.



Advice and Tips

A is used instead of y here. Differentiate in the usual way.

a Explain why, if one side of the rectangle is x cm, the other will be $(15 - x)$ cm.

Suppose the area of the rectangle is A cm^2 then $A = x(15 - x)$.

b Find $\frac{dA}{dx}$.

c Find the turning point of the graph $A = x(15 - x)$.

d Is the turning point a maximum or a minimum?

e What do the answers to **c** and **d** tell you about the area of the rectangle?

Check your progress

Extended

- I understand the idea of a derivative of a function
- I can find the derivatives of functions of the form ax^n and simple sums of these where $n = 0, 1, 2, 3$ or 4
- I can apply differentiation to find gradients
- I can apply differentiation to find turning points
- I can discriminate between maximum points and minimum points

Examination questions: Algebra

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PAPER 1

CORE

1 Simplify $\frac{r^6}{r^2}$. [1]

Cambridge International IGCSE Mathematics 0580 *Paper 11 Q3 Oct/Nov 2015*

2 **a** $s = 4t + 3u$

Calculate s when $t = 2.6$ and $u = -0.4$. [2]

b Solve $5x - 7 = 10$. [2]

Cambridge International IGCSE Mathematics 0580 *Paper 11 Q19 Oct/Nov 2015*

3 **a** Write down the next term in each of these sequences.

i 5 9 13 17 ... [1]

ii 3 6 12 24 ... [1]

b Here are the first four terms in a different sequence.

2 7 12 17

Find an expression for the nth term of this sequence. [2]

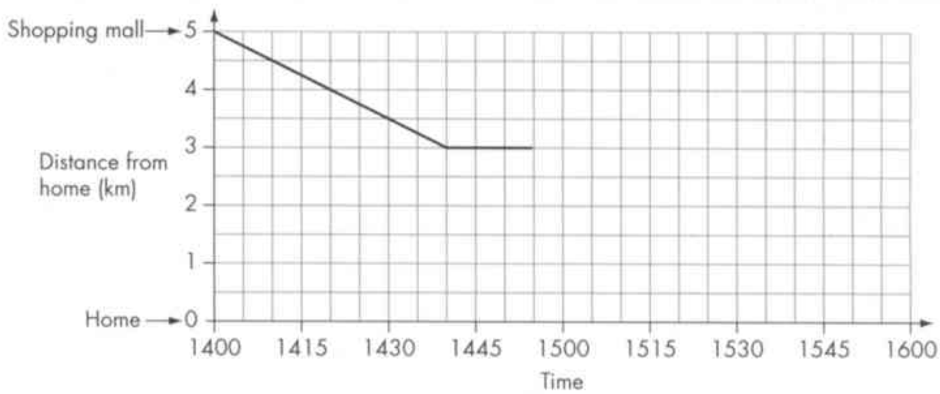
Cambridge International IGCSE Mathematics 0580 *Paper 11 Q22 Oct/Nov 2015*

4 **a** Maria travels by bus to the shopping mall.

She leaves home at 11 50 and arrives at the shopping mall at 12 17.

How many minutes does it take Maria to travel from home to the shopping mall? [1]

b



Maria walks home from the shopping mall.

The travel graph shows part of her journey.

i Maria stops at her friend's house on the way home.

How far from the shopping mall does her friend live? [1]

ii Maria leaves her friend's house at 1455.

She walks the rest of the way home at a constant speed of 4 km/h.

Complete the travel graph. [2]

Cambridge International IGCSE Mathematics 0580 *Paper 11 Q20 Oct/Nov 2015*

5 Simplify.

$6uw^{-3} \times 4uw^{6}$ [2]

Cambridge International IGCSE Mathematics 0580 *Paper 11 Q10 May/June 2015*

6 **a** Factorise.

$3w^2 - 2w$ [1]

b Expand and simplify.

$x(2x + 3) + 5(x - 7)$ [2]

Cambridge International IGCSE Mathematics 0580 *Paper 11 Q13 May/June 2015*

7 Solve the equation.

$\frac{n-8}{2} = 11$ [2]

Cambridge International IGCSE Mathematics 0580 *Paper 11 Q6 May/June 2014*

8 Solve the simultaneous equations.

$5x + 2y = 16$

$3x - 4y = 7$ [3]

Cambridge International IGCSE Mathematics 0580 *Paper 11 Q20 Oct/Nov 2014*

9 **a** Find the value of $5x^2$ when $x = -4$. [2]

b Make x the subject of the formula $y = 5x^2$. [2]

Cambridge International IGCSE Mathematics 0580 *Paper 11 Q21 Oct/Nov 2014*

10 **a** Simplify the expressions.

i $p^3 \times p^7$ [1]

ii $t^5 \div t^8$ [1]

b $(h^3)^k = h^{12}$

Find the value of k. [1]

Cambridge International IGCSE Mathematics 0580 *Paper 11 Q16 May/June 2014*

11 Rearrange this equation to make b the subject.

$a = \frac{b}{5} - 9$ [2]

Cambridge International IGCSE Mathematics 0580 *Paper 11 Q9 May/June 2013*

12 Here are the first four terms of a sequence.

4 11 18 25

Write down an expression for the nth term. [2]

Cambridge International IGCSE Mathematics 0580 *Paper 11 Q10 May/June 2013*

CORE

13 **a** Which **two** of these have the same value?

5^{-2} $\quad \frac{2}{5}$ $\quad \left(\frac{1}{2}\right)^2$ $\quad \left(\frac{2}{5}\right)^2$ $\quad 0.2^2$ [2]

b Simplify.

i $a^6 \times a^3$ [1]

ii $24b^{16} \div 6b^4$ [2]

Cambridge International IGCSE Mathematics 0580 *Paper 11 Q18 May/June 2013*

14 **a** Multiply out the brackets.

$5(x + 3)$ [1]

b Factorise completely.

$12xy - 3x^2$ [2]

c Solve.

$5x - 24 = 51$ [2]

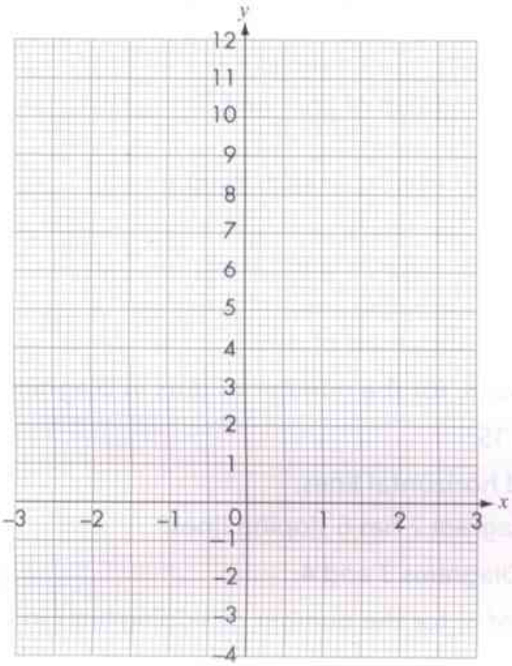
Cambridge International IGCSE Mathematics 0580 *Paper 11 Q19 May/June 2013*

PAPER 3

1 **a** **i** Complete the table of values for $y = 8 - x^2$. [2]

x	−3	−2	−1	0	1	2	3
y	−1			8	7		−1

ii On the grid, draw the graph of $y = 8 - x^2$ for $-3 \leqslant x \leqslant 3$. [4]



iii Write down the equation of the line of symmetry of the graph. [1]

iv Use your graph to solve the equation $8 - x^2 = 0$. [2]

b **i** On the grid, plot the points (−2, 8) and (2.5, −1).

Draw a straight line through these points. [2]

ii Find the equation of your line in the form $y = mx + c$. [3]

iii Write down the co-ordinates of the point of intersection of your line with $y = 8 - x^2$. [1]

Cambridge International IGCSE Mathematics 0580 *Paper 31 Q6 Oct/Nov 2014*

2 **a** In 2001 Arnold was x years old.

Ken is **34 years younger** than Arnold.

i Complete the table, in terms of x, for Arnold's and Ken's ages. [3]

	2001	2013
Arnold's age	x	
Ken's age		

ii In 2013 Arnold is **three** times as old as Ken.

Write down an equation in x and solve it. [4]

CORE

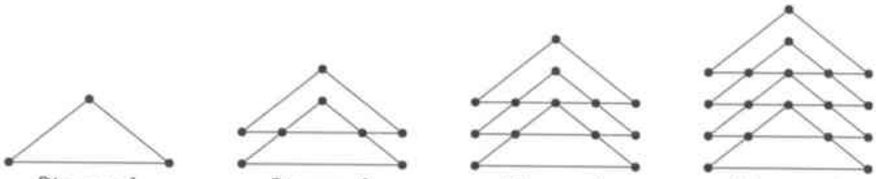
b Solve the simultaneous equations.

$3x + 2y = 18$

$2x - y = 19$ [3]

Cambridge International IGCSE Mathematics 0580 *Paper 31 Q10 May/June 2013*

3



Diagrams 1 to 4 show a sequence of shapes made up of lines and dots at the intersections of lines.

a i Complete the table showing the number of dots in each diagram. [3]

Diagram	1	2	3	4	5	6
Dots	3	8	13			

ii Write down the rule for continuing the sequence of dots. [1]

iii Write down an expression, in terms of n, for the number of dots in Diagram n. [2]

iv Find the number of dots in Diagram 15. [1]

b The dots are joined by sloping lines and horizontal lines.

i Diagram 1 has 2 sloping lines and Diagram 2 has 6 sloping lines.

Find the number of sloping lines in Diagrams 3 and 4. [2]

ii Write down an expression, in terms of n, for the number of sloping lines in Diagram n. [2]

Cambridge International IGCSE Mathematics 0580 *Paper 31 Q9 Oct/Nov 2014*

4 **a** The cost, $\$C$, of hiring a meeting room for n people is calculated using the formula $C = 80 + 5n$.

i Calculate C when $n = 12$. [2]

ii Maria pays \$230 to hire the meeting room.

Work out the number of people at the meeting. [2]

iii Make n the subject of the formula $C = 80 + 5n$. [2]

b Expand and simplify $2(3x + 4) - 3(2 - x)$. [2]

c Solve the simultaneous equations.

$3x + y = 13$

$2x + 3y = 18$ [3]

Cambridge International IGCSE Mathematics 0580 *Paper 31 Qa Oct/Nov 2012*

5 **a** Simplify $x^8 \div x^2$. [1]

b Simplify $\left(\frac{x^6}{27}\right)^{\frac{1}{3}}$. [2]

Cambridge International IGCSE Mathematics 0580 *Paper 21 Q11 Oct/Nov 2014*

6 Make x the subject of the formula.

$y = (x - 4)^2 + 6$ [3]

Cambridge International IGCSE Mathematics 0580 *Paper 21 Q7 May/June 2014*

Examination questions: Algebra

PAPER 2

EXTENDED

1 Factorise completely.

a $ax + ay + 3cx + 3cy$ [2]

b $3a^2 - 12b^2$ [3]

Cambridge International IGCSE Mathematics 0580 *Paper 21 Q9 Oct/Nov 2015*

2 Simplify.

$$\frac{x^2 - 16}{x^2 - 3x - 4}$$ [4]

Cambridge International IGCSE Mathematics 0580 *Paper 21 Q15 Oct/Nov 2015*

3 $f(x) = x^3$ $g(x) = 3x - 5$ $h(x) = 2x + 1$

Work out

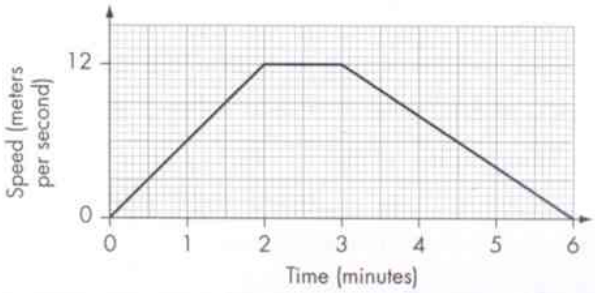
a ff(2), [2]

b gh(x) and simplify your answer, [2]

c $h^{-1}(x)$, the inverse of h(x). [2]

Cambridge International IGCSE Mathematics 0580 *Paper 21 Q21 Oct/Nov 2015*

4



A tram leaves a station and accelerates for 2 **minutes** until it reaches a speed of 12 metres per second.

It continues at this speed for 1 minute.

It then decelerates for 3 minutes until it stops at the next station.

The diagram shows the speed-time graph for this journey.

Calculate the distance, in metres, between the two stations. [3]

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5 Find the nth term of each sequence.

a 4, 8, 12, 16, 20, ... [1]

b 11, 20, 35, 56, 83, ... [2]

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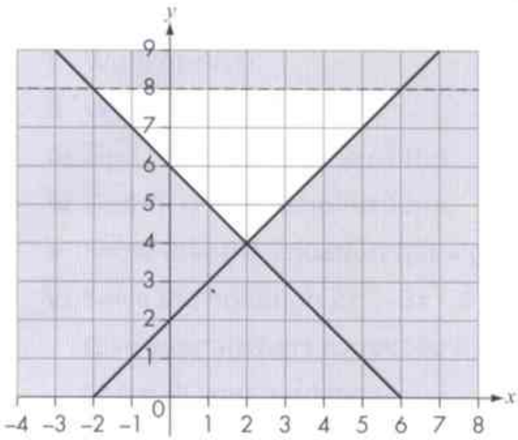
6



Write down the 3 inequalities which define the unshaded region. [4]

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7 Factorise completely.

a $yp + yt + 2xp + 2xt$ [2]

b $7(h + k)^2 - 21(h + k)$ [2]

Cambridge International IGCSE Mathematics 0580 *Paper 21 Q20 May/June 2015*

8 Solve the simultaneous equations.

$0.4x - 5y = 27$

$2x + 0.2y = 9$ [3]

Cambridge International IGCSE Mathematics 0580 *Paper 21 Q12 Oct/Nov 2014*

9 Write as a single fraction in its simplest form.

$\frac{2}{x} - \frac{2}{x+1}$ [3]

Cambridge International IGCSE Mathematics 0580 *Paper 21 Q8 May/June 2014*

10 Factorise completely.

a $ax + ay + bx + by$ [2]

b $3(x - 1)^2 + (x - 1)$ [2]

Cambridge International IGCSE Mathematics 0580 *Paper 21 Q10 May/June 2014*

11 Solve the inequality for positive integer values of x.

$\frac{21 + x}{5} > x + 1$ [4]

Cambridge International IGCSE Mathematics 0580 *Paper 21 Q15 May/June 2014*

12 **a** $(2^{24})^{\frac{1}{2}} = p^4$

Find the value of p. [2]

b Simplify $\frac{q^2 + q^2}{q^{\frac{1}{4}} \times q^{\frac{1}{4}}}$. [3]

Cambridge International IGCSE Mathematics 0580 *Paper 21 Q16 may/June 2014*

PAPER 4

EXTENDED

1 **a** Calculate $2^{0.7}$. [1]

b Find the value of x in each of the following.

i $2x = 128$ [1]

ii $2x \times 2^9 = 2^{13}$ [1]

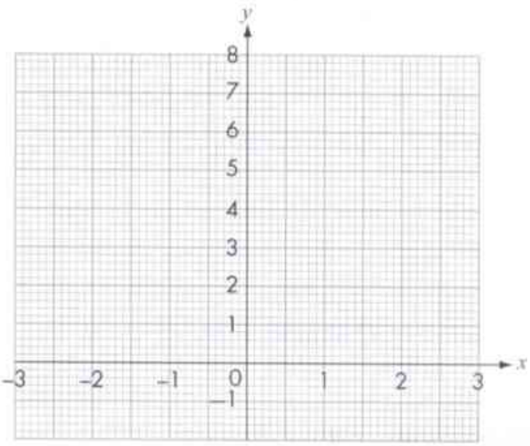
iii $2^9 \div 2^x = 4$ [1]

iv $2^x = \sqrt[3]{2}$ [1]

c **i** Complete this table of values for $y = 2^x$. [2]

x	−3	−2	−1	0	1	2	3
y	0.125		0.5		2	4	8

ii On the grid, draw the graph of $y = 2^x$ for $-3 \leqslant x \leqslant 3$. [4]



iii Use your graph to solve $2^x = 5$. [1]

iv Find the equation of the line joining the points (1, 2) and (3, 8). [3]

v By drawing a suitable line on your graph, solve $2^x - 2 - x = 0$. [2]

Cambridge International IGCSE Mathematics 0580 *Paper 41 Q2 Oct/Nov 2015*

2 **a** Factorise $x^2 - 3x - 10$. [2]

b **i** Show that $\frac{x+2}{x+1} + \frac{3}{x} = 3$ simplifies to $2x^2 - 2x - 3 = 0$. [3]

ii Solve $2x^2 - 2x - 3 = 0$.

Give your answers correct to 3 decimal places.

Show all your working. [4]

c Simplify $\frac{2x+3}{x+2} - \frac{x}{x+1}$. [4]

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3 **a** $f(x) = 2x - 3 \quad g(x) = \frac{1}{x+1} + 2 \quad h(x) = 3^x$

i Work out f(4). [1]

ii Work out fh(–l). [2]

iii Find $f^{-1}(x)$, the inverse of $f(x)$. [2]

iv Find $ff(x)$ in its simplest form. [2]

v Show that the equation $f(x) = g(x)$ simplifies to $2x^2 - 3x - 6 = 0$. [3]

vi Solve the èquation $2x^2 - 3x - 6 = 0$.

Give your answers correct to 2 decimal places.

Show all your working. [4]

b Simplify $\frac{x^2 - 3x + 2}{x^2 + 3x - 10}$. [4]

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4

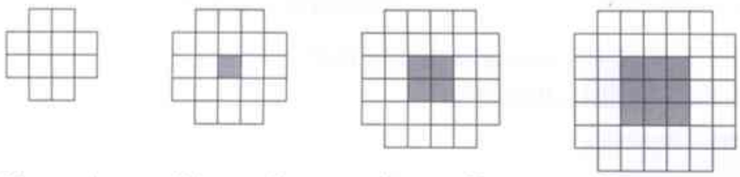

Diagram 1 Diagram 2 Diagram 3 Diagram 4

The first four diagrams in a sequence are shown above.

The diagrams are drawn using white squares ☐ and grey squares ■.

a Complete the columns in the table for Diagram 4 and Diagram n. [6]

Diagram	1	2	3	4	n
Number of white squares	12	20	28		
Number of grey squares	0	1	4		
Total number of squares	12	21	32		$(n + 1)(n + 5)$

b Work out the number of the diagram which has a total of 480 squares. [2]

c The **total** number of squares in the **first n diagrams** is $\frac{1}{3}n^3 + pn^2 + qn$.

i Use $n = 1$ in this expression to show that $p + q = 11\frac{2}{3}$. [1]

ii Use $n = 2$ in the expression to show that $4p + 2q = 30\frac{1}{3}$. [2]

iii Find the values of p and q. [3]

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Chapter 23

Angle properties

Topics	Level	Key words
1 Angle facts	CORE	angles at a point, angles on straight lines, opposite angles, vertically opposite angles
2 Parallel lines	CORE	corresponding angles, alternate angles, allied angles, interior angles
3 Angles in a triangle	CORE	equilateral triangle, isosceles triangle, right-angled triangle
4 Angles in a quadrilateral	CORE	quadrilateral, parallelogram, rhombus, kite, trapezium
5 Regular polygons	CORE	polygon, regular polygon, external angles, pentagon, hexagon, octagon, square
6 Irregular polygons	EXTENDED	irregular polygon, heptagon, nonagon, decagon
7 Tangents and diameters	CORE	radius, tangent, point of contact, diameter, semi-circle
8 Angles in a circle	EXTENDED	arc, circumference, segment, subtended
9 Cyclic quadrilaterals	EXTENDED	cyclic quadrilateral, opposite segment, supplementary
10 Alternate segment theorem	EXTENDED	tangent, chord, major segment, minor segment, major arc, minor arc, alternate segment

In this chapter you will learn how to:

CORE

- **Calculate unknown angles using the following geometrical properties:**
  - (a) angles at a point
  - (b) angles at a point on a straight line and intersecting straight lines
  - (c) angles formed within parallel lines
  - (d) angle properties of triangles and quadrilaterals
  - (e) angle properties of regular polygons
  - (f) angle in a semi-circle
  - (g) angle between tangent and radius of a circle. (C4.7 and E4.7)

EXTENDED

- **Use in addition the following geometrical properties:**
  - (a) angle properties of irregular polygons
  - (b) angle at the centre of a circle is twice the angle at the circumference
  - (c) angles in the same segment are equal
  - (d) angles in opposite segments are supplementary; cyclic quadrilaterals
  - (e) alternate segment theorem. (E4.7)

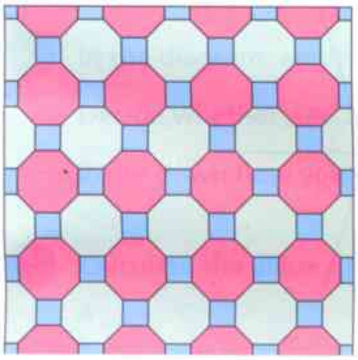
Why this chapter matters

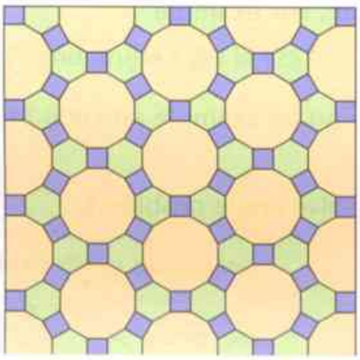
Angles describe an amount of turn around a point. It is important to be able to measure them and understand their properties.

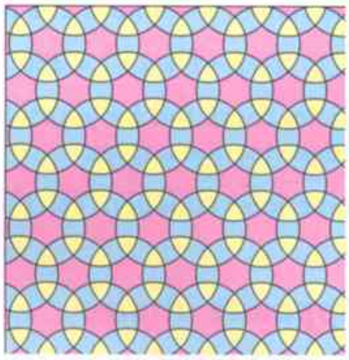
In a regular polygon all the angles are the same size and all the sides are the same length. The shape of the polygon depends on how many angles and sides it has:

- a triangle has 3 sides and 3 angles
- a square has 4 sides and 4 angles
- a pentagon has 5 sides and 5 angles
- a hexagon has 6 sides and 6 angles.

The patterns below are made by putting together different regular polygons. How many different polygons can you see in them? Can you see any octagons (8-sided shapes) or decagons (10-sided shapes)?







Some shapes fit together better than others because of the size of their angles.

Bees construct their hives from hexagon shapes, which can join together without gaps.

Squares and rectangles also fit together easily, which makes them an ideal shape to use in building.

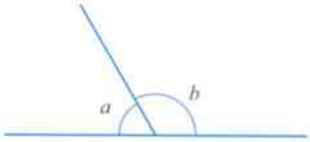
In this chapter you will look at angles, the shapes they form, and their properties.

Chapter 23 . Topic 1

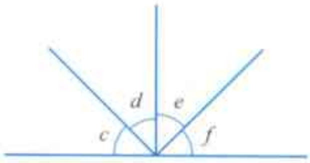
23.1 Angle facts

Angles on a line

The **angles on a straight line** add up to 180°.



$a + b = 180°$



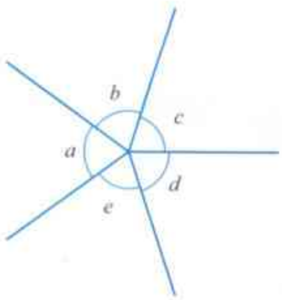
$c + d + e + f = 180°$

Draw an example for yourself (and measure all the angles) to show that the statement is true.

Angles at a point

The sum of the **angles at a point** is 360°. For example:

$$a + b + c + d + e = 360°$$



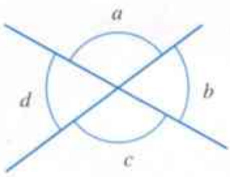
Again, check this for yourself by drawing an example and measuring the angles.

Sometimes equations can be used to solve angle problems.

Opposite angles

Opposite angles are equal.

So $a = c$ and $b = d$.



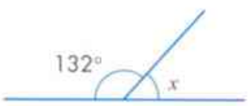
Sometimes opposite angles are called **vertically opposite angles**.

EXERCISE 23A

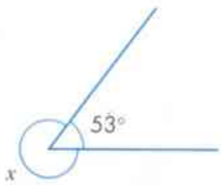
CORE

1 Calculate the value of x in each diagrams.

a



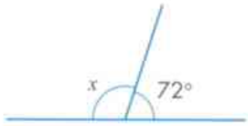
b



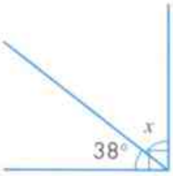
Advice and Tips

Never measure angles in questions like these. Diagrams in textbooks are not usually drawn accurately. Always calculate angles unless you are told to measure them.

c

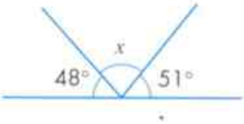


d

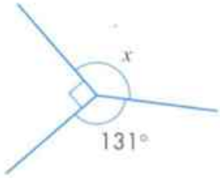


e

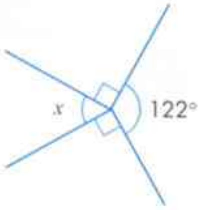
f



g

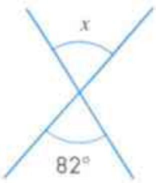


h



2 Write down the value of x in each of these diagrams.

a



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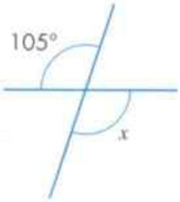

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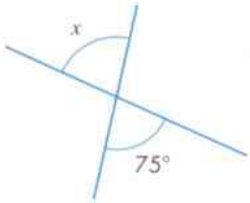
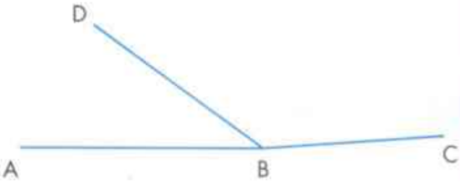


3 In the diagram, angle ABD is 45° and angle CBD is 125°.

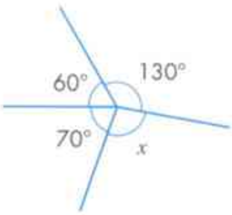
Decide whether ABC is a straight line.

Write down how you decided.



4 Calculate the value of x in each of these diagrams.

a



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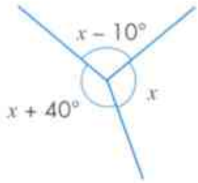

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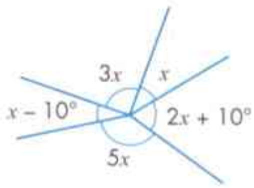
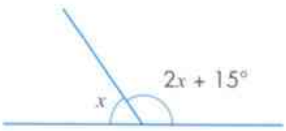


5 Calculate the value of x in each of these diagrams.

a



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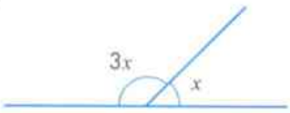

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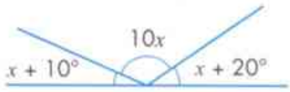
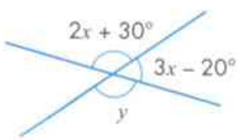
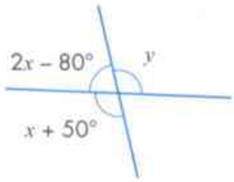


6 Calculate the value of x first and then calculate the value of y in each of these diagrams.

a



b



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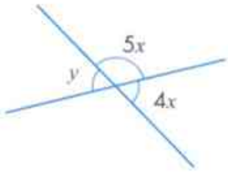
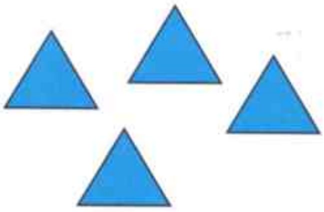


CORE

7 Shalini has a collection of tiles.
They are all equilateral triangles and are all the same size.



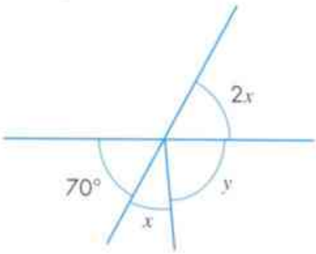
She says that six of the tiles will fit together and leave no gaps.

Explain why Shalini is correct.

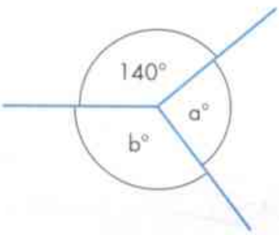
Advice and Tips

All the angles in an equilateral triangle are 60°.

8 Work out the value of y in the diagram.



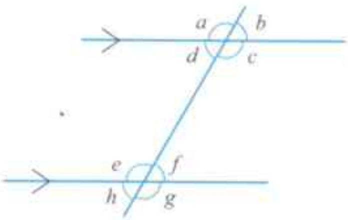
9 The ratio $a : b = 2 : 3$



Work out the values of a and b.

23.2 Parallel lines

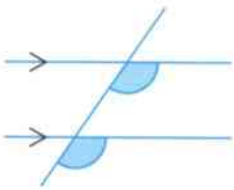
Angles in parallel lines



The arrowheads indicate that the lines are parallel and the line that crosses the parallel lines is called a transversal.

Notice that eight angles are formed.

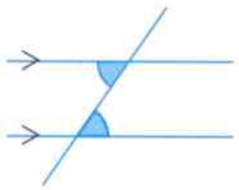
Angles like these



are called **corresponding angles** (Look for the letter F).

Corresponding angles are equal.

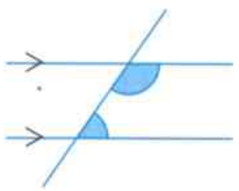
Angles like these



are called **alternate angles** (Look for the letter Z).

Alternate angles are equal.

Angles like these

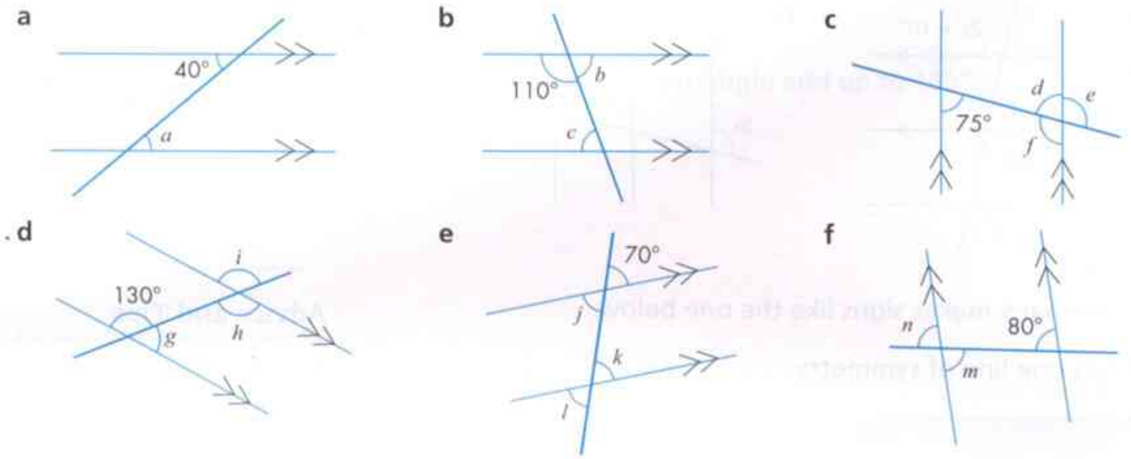


are called **allied angles** or **interior angles** (Look for the letter C).

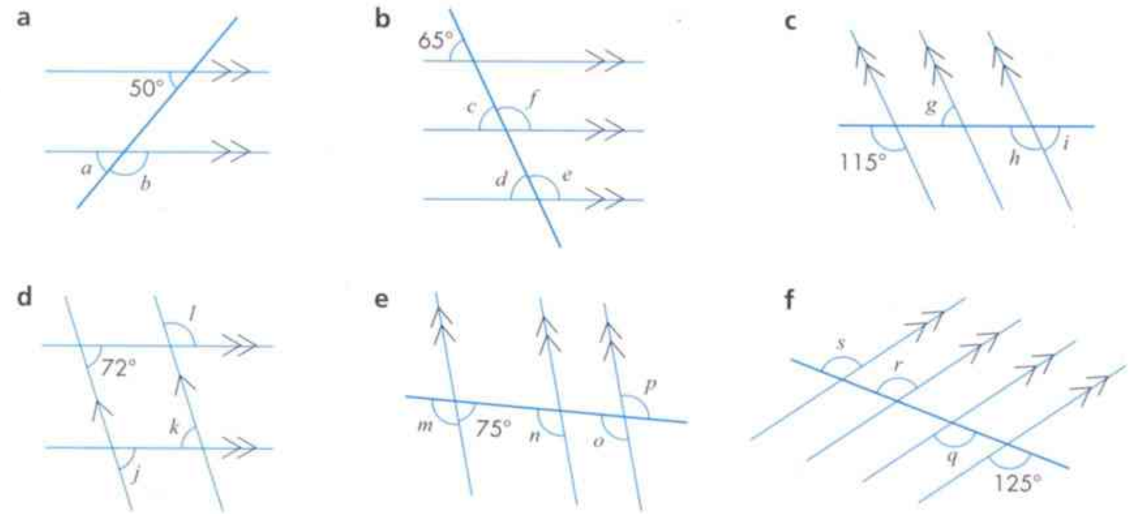
Allied angles add to 180°.

EXERCISE 23B

1 State the sizes of the lettered angles in each diagram.

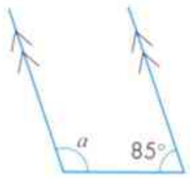


2 State the sizes of the lettered angles in each diagram.

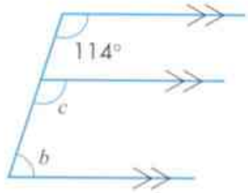


3 State the sizes of the lettered angles in these diagrams.

a

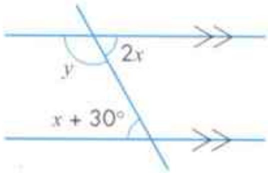


b

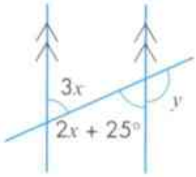


4 Calculate the values of x and y in these diagrams.

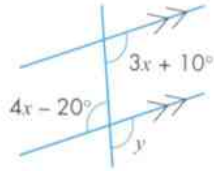
a



b

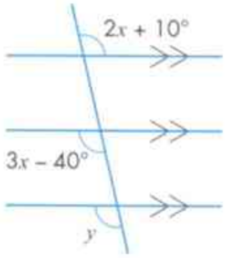


c

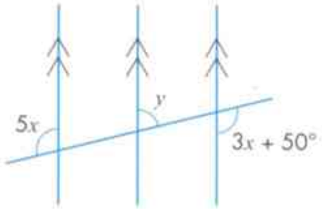


5 Calculate the values of x and y in these diagrams.

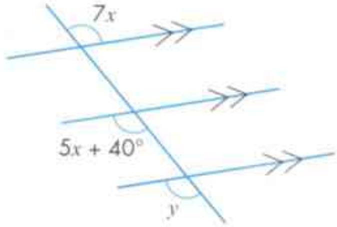
a



b

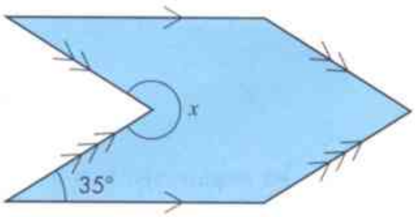


c



6 A company makes signs like the one below.

It has one line of symmetry.



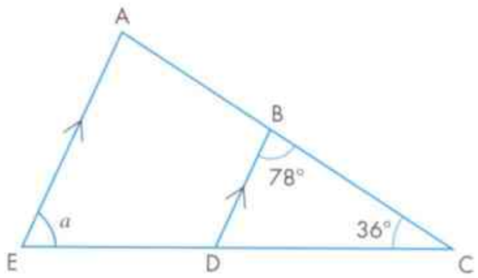
The company needs to know the size of the angle marked x on the diagram.

Work out the size of the angle labelled x.

Advice and Tips

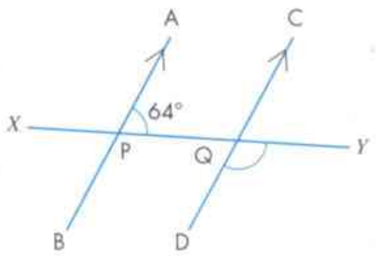
Draw the line of symmetry on the shape first.

7 In the diagram, AE is parallel to BD.



Work out the size of the angle labelled a.

8



The line XY crosses the parallel lines AB and CD at P and Q.

a Work out the size of angle DQY. Give reasons for your answer.

b This is Vreni's solution.

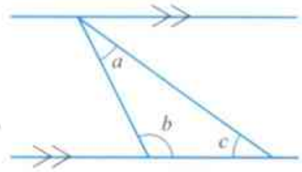
Angle $PQD = 64°$ (corresponding angles)

So angle $DQY = 124°$ (angles on a line = 190°)

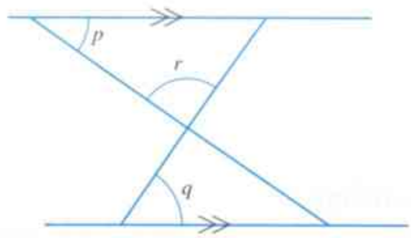
She has made a number of errors in her solution.

Write out the correct solution for the question.

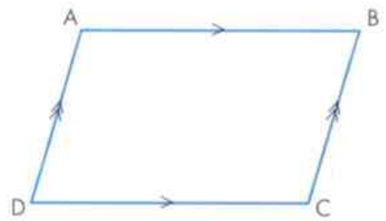
9 Use the diagram to prove that the three angles in a triangle add up to 180°.



10 Prove that $p + q + r = 180°$.



11



$ABCD$ is a parallelogram.

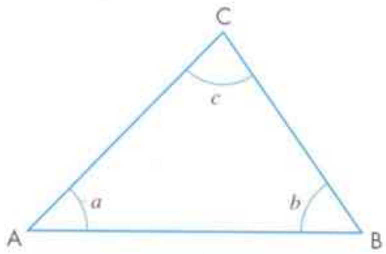
AB is parallel to CD and DA is parallel to CB.

Prove that the opposing angles of the parallelogram are the same size.

Chapter 23 . Topic 3

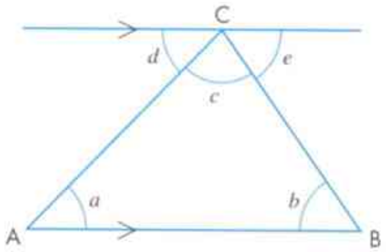
23.3 Angles in a triangle

The three angles in any triangle add up to 180°.



$a + b + c = 180°$

You can prove this by drawing a line through C parallel to AB.

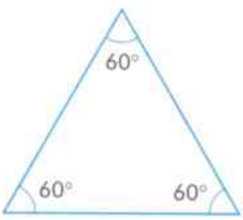


$a = d$	(alternate angles)
$b = e$	(alternate angles)
$d + e + c = 180°$	(angles on a straight line)

So $a + b + c = 180°$

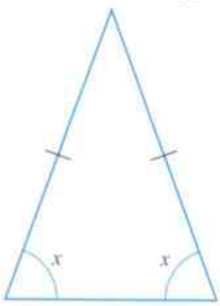
Special triangles

Equilateral triangle



An **equilateral triangle** is a triangle with all its sides equal. Therefore, all three interior angles are 60°.

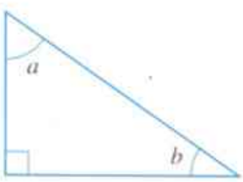
Isosceles triangle



An **isosceles triangle** is a triangle with two equal sides and, therefore, with two equal interior angles (at the foot of the equal sides).

Notice how to mark the equal sides and equal angles.

Right-angled triangle

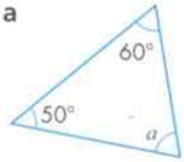


A **right-angled triangle** has an interior angle of 90°.
$a + b = 90°$

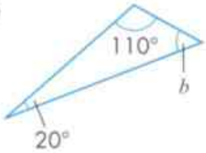
EXERCISE 23C

1 Find the size of the angle marked with a letter in each of these triangles.

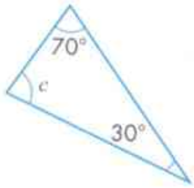
a



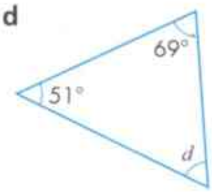
b



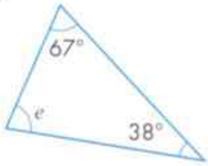
c



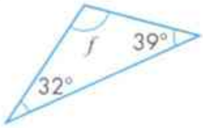
d



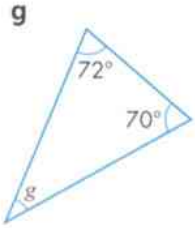
e



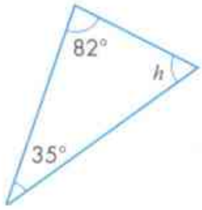
f



g



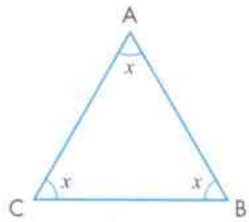
h



2 Do any of these sets of angles form the three angles of a triangle? Explain your answer.

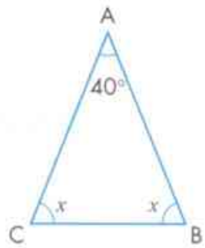
- **a** 35°, 75°, 80°
- **b** 50°, 60°, 70°
- **c** 55°, 55°, 60°
- **d** 60°, 60°, 60°
- **e** 35°, 35°, 110°
- **f** 102°, 38°, 30°

3 In the triangle on the right, all the interior angles are the same.



- **a** What is the size of each angle?
- **b** What is the name of a special triangle like this?
- **c** What is special about the sides of this triangle?

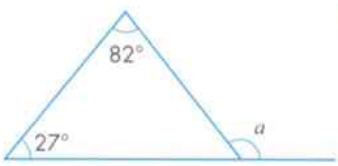
4 In the triangle on the right, two of the angles are the same.



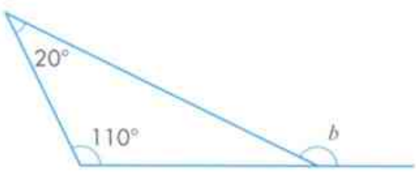
- **a** Work out the size of the lettered angles.
- **b** What is the name of a special triangle like this?
- **c** What is special about the sides AC and AB of this triangle?

5 Find the size of the angle marked with a letter in each of these diagrams.

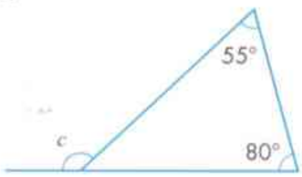
a



b



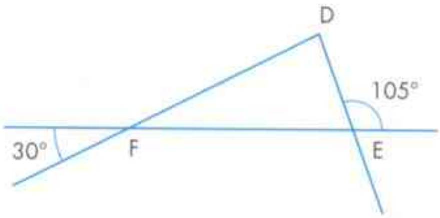
c



CORE

6 What is the special name for triangle DEF?

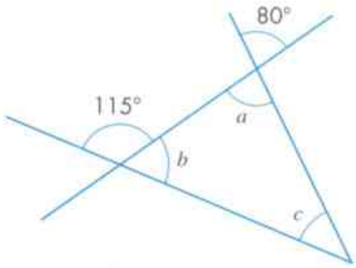
Show all your working to explain your answer.



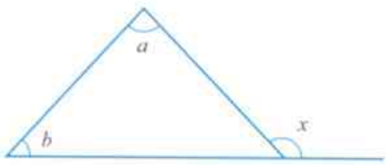
7 The diagram shows three intersecting straight lines.

Work out the values of a, b and c.

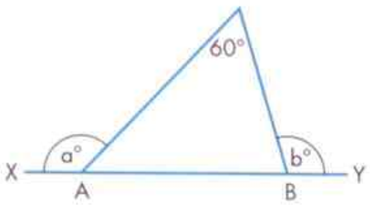
Give reasons for your answers.



8 By using algebra, show that $x = a + b$.



9



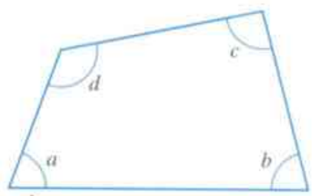
ABC is a triangle and $XABY$ is a straight line.

Find a formula for b in terms of a.

23.4 Angles in a quadrilateral

The four angles in any **quadrilateral** add up to 360°.

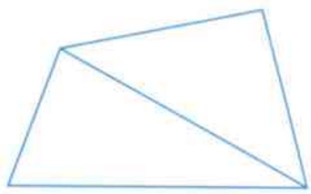
$a + b + c + d = 360°$



You can check this by dividing the quadrilateral into two triangles.

The six angles of the triangles are the same as the four angles of the quadrilateral.

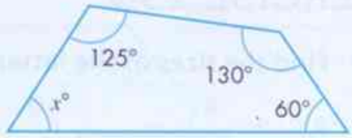
Sum of angles of a quadrilateral = 180° + 180° = 360°



Example 1

Three angles of a quadrilateral are 125°, 130° and 60°.

Find the size of the fourth angle.



$125 + 130 + 60 + x = 360$

$315 + x = 360$

$x = 360 - 315$

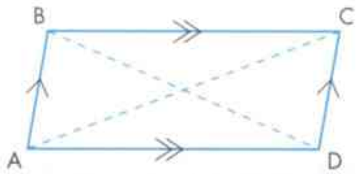
$x = 45$

So the fourth angle is 45°.

Special quadrilaterals

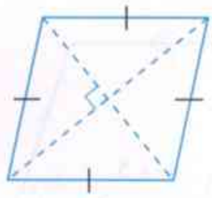
A **parallelogram** has opposite sides that are parallel.

Its opposite sides are equal. Its diagonals bisect each other.
Its opposite angles are equal: that is, angle A = angle C and angle B = angle D



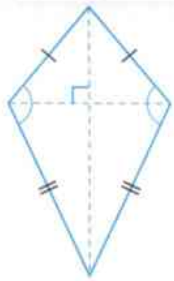
A **rhombus** is a parallelogram with all its sides equal.

Its diagonals bisect each other at right angles.
Its diagonals also bisect the angles at the vertices.



A **kite** is a quadrilateral with two pairs of equal adjacent sides.

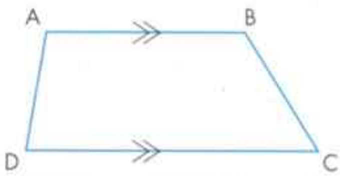
Its longer diagonal bisects its shorter diagonal at right angles.
The opposite angles between the sides of different lengths are equal.



A **trapezium** has two parallel sides.

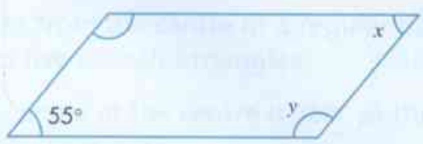
The sum of the interior angles at the ends of each non-parallel side is 180°.

angle A + angle D = 180° and angle B + angle C = 180°



Example 2

Find the size of the angles marked x and y in this parallelogram.



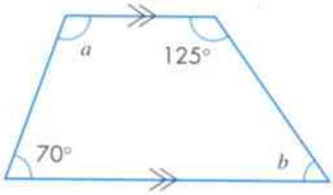
$x = 55°$ (opposite angles are equal) and $y = 125°$ ($x + y = 180°$)

EXERCISE 23D

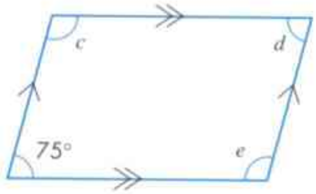
CORE

1 Find the sizes of the lettered angles in these quadrilaterals.

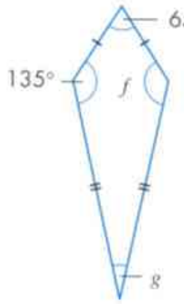
a



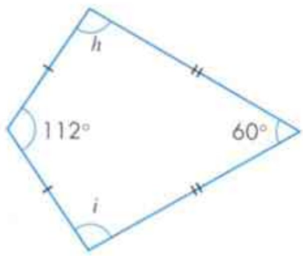
b



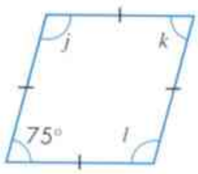
c



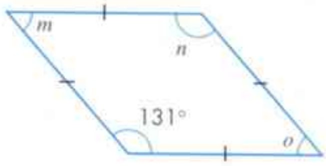
d



e

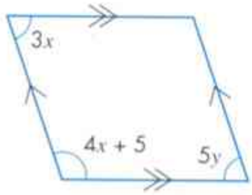


f

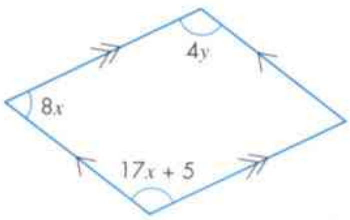


2 Calculate the values of x and y in each of these quadrilaterals.

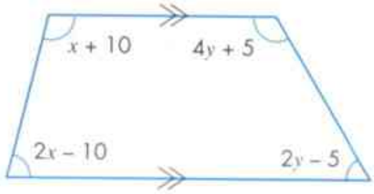
a



b



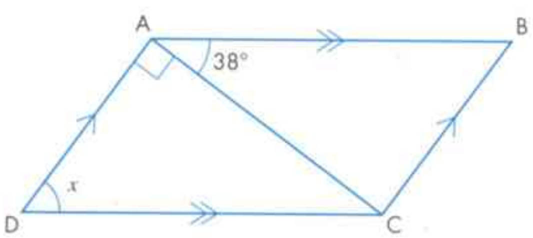
c



3 Find the value of x in each of these quadrilaterals and state what type of quadrilateral it could be. All angles are in degrees.

- **a** A quadrilateral with angles $x + 10$, $x + 20$, $2x + 20$, $2x + 10$
- **b** A quadrilateral with angles $x - 10$, $2x + 10$, $x - 10$, $2x + 10$
- **c** A quadrilateral with angles $x - 10$, $2x$, $5x - 10$, $5x - 10$
- **d** A quadrilateral with angles $4x + 10$, $5x - 10$, $3x + 30$, $2x + 50$

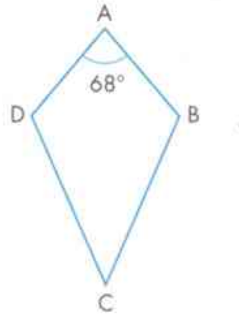
4 The diagram shows a parallelogram ABCD.



Work out the value of x, marked on the diagram.

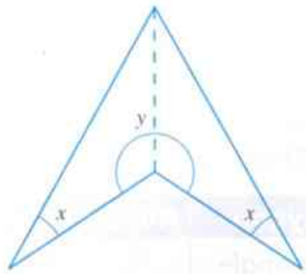
5 Dani is making a kite and wants angle C to be half of angle A.

Work out the size of angles B and D.



6 This quadrilateral is made from two isosceles triangles. They are both the same size.

Find the value of y in terms of x.



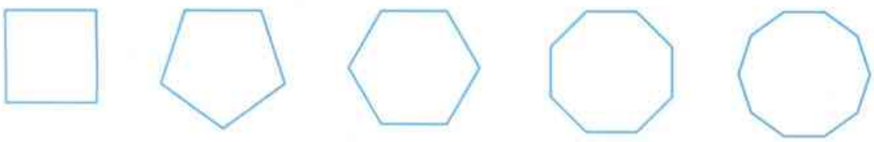
7 The four angles of a quadrilateral are in the ratio 1 : 2 : 3 : 4

Calculate the four angles.

23.5 Regular polygons

Regular polygons

Here are five **regular polygons**.

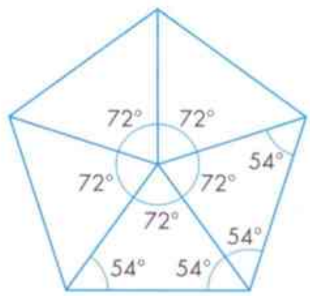


A **polygon** is regular if all its interior angles are equal and all its sides are the same length.

A **square** is a regular four-sided shape that has an angle sum of 360°.

So, each angle is 360° ÷ 4 = 90°.

The angles of a regular polygon



Lines from the centre of a regular pentagon divide it into five isosceles triangles.

The angle at the centre is 360° so the angle of each isosceles triangle at the centre is:

360° ÷ 5 = 72°

The other angles in each triangle are identical so each one is:

$(180 - 72) \div 2 = 54°$

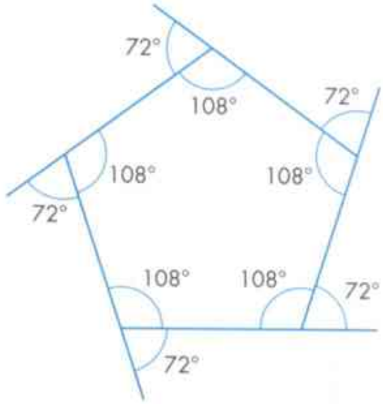
So each interior angle of a regular pentagon is:

$2 \times 54° = 108°$

There is also an **exterior angle** at each vertex. It is:

$180 - 108 = 72°$

Notice that $72° = 360° \div 5$.



General result

If a regular polygon has n sides each exterior angle is $\frac{360°}{n}$.

If you want to find the interior angle of a regular polygon it may be easier to find the exterior angle like this first. Then subtract it from 180° to find the interior angle.

Regular polygon	Number of sides	Exterior angle	Interior angle
Equilateral triangle	3	120°	60°
Square	4	90°	90°
Pentagon	5	72°	108°
Hexagon	6	60°	120°
Octagon	8	45°	135°

Example 3

Calculate the size of the exterior and interior angle for a regular 12-sided polygon (a regular dodecagon).

Exterior angle $= \frac{360°}{12} = 30°$

interior angle $= 180° - 30° = 150°$

EXERCISE 23E

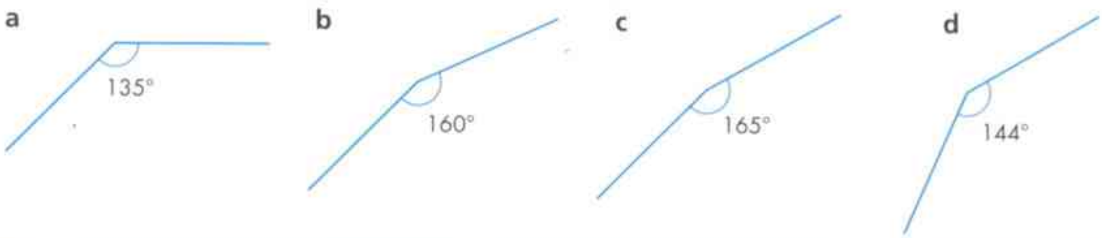
CORE

1 Each diagram shows an interior angle of a regular polygon. For each polygon, answer these questions.

i What is the size of its exterior angle?

ii How many sides does it have?

iii What is the sum of its interior angles?



2 Each diagram shows an exterior angle of a regular polygon. For each polygon, answer these questions.

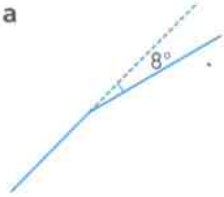
i What is the size of its interior angle?

ii How many sides does it have?

iii What is the sum of its interior angles?

Advice and Tips

Remember that the angle sum is calculated as (number of sides – 2) × 180°.

a


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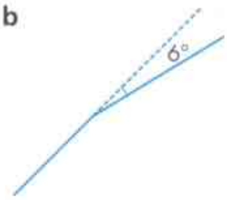
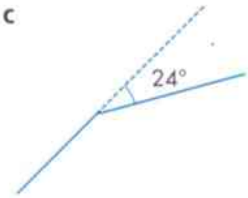
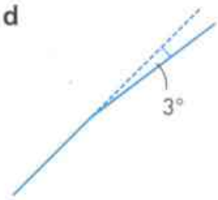
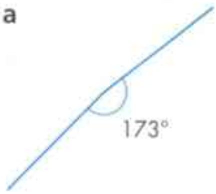


c


d


3 None of these angles can be the interior angle of a regular polygon. Explain why.

a


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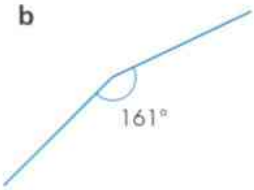

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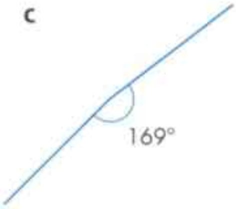

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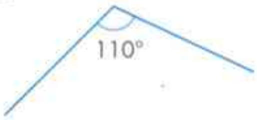
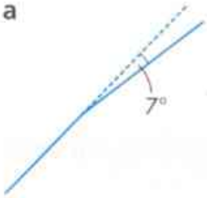


4 None of these angles can be the exterior angle of a regular polygon. Explain why.

a


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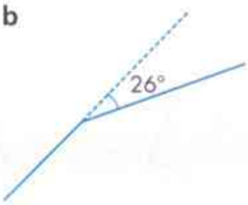

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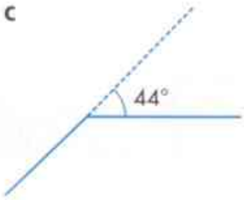

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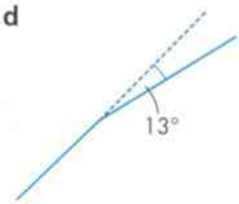
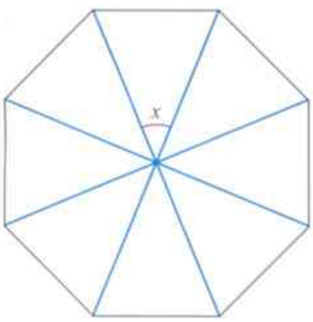


5 Draw a sketch of a regular octagon and join each vertex to the centre.



Calculate the value of the angle at the centre (marked x).

What connection does this have with the exterior angle?

Is this true for all regular polygons?

6 The diagram shows part of a regular polygon.

Each interior angle is 144°.

a What is the size of each exterior angle of the polygon?

b How many sides does the polygon have?

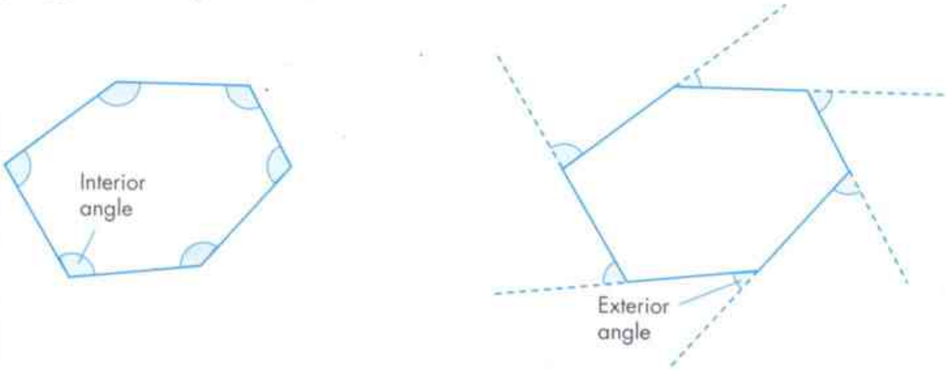
7 a Show that it is possible to draw a regular polygon with an interior angle of 170°.

b Show that it is not possible to draw a regular polygon with an interior angle of 169°.

23.6 Irregular polygons

E

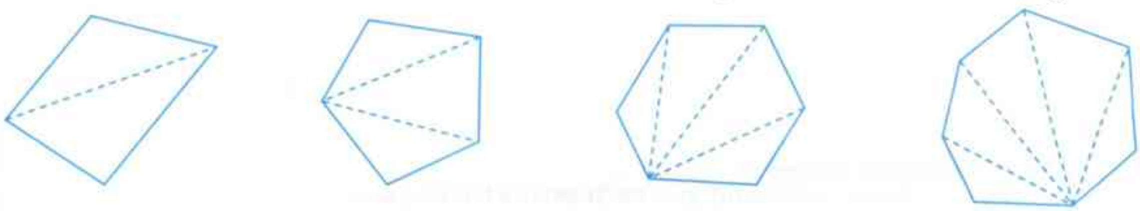
A polygon is regular if all its sides are the same length and all its angles are the same size. Any other polygon is **irregular**.



The *exterior* angles of *any* polygon add up to 360°.

Interior angles

You can find the sum of the interior angles of any polygon by splitting it into triangles.



Since you already know that the angles in a triangle add up to 180°, you find the sum of the interior angles in a polygon by multiplying the number of triangles in the polygon by 180°, as shown in this table.

Shape	Name	Sum of interior angles
4-sided	**Quadrilateral**	$2 \times 180° = 360°$
5-sided	**Pentagon**	$3 \times 180° = 540°$
6-sided	**Hexagon**	$4 \times 180° = 720°$
7-sided	**Heptagon**	$5 \times 180° = 900°$
8-sided	**Octagon**	$6 \times 180° = 1080°$
9-sided	**Nonagon**	$7 \times 180° = 1260°$
10-sided	**Decagon**	$8 \times 180° = 1440°$

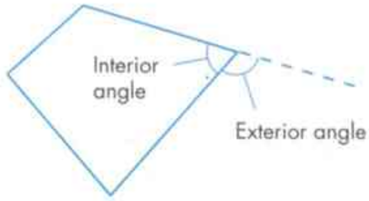
As you can see from the table, for an n-sided polygon, the sum of the interior angles, S, is given by the formula:

$$S = 180(n - 2)°$$

Exterior angles

As in regular polygons the sum of all the exterior angles in an irregular polygon is 360° but their sizes may not be the same.

The size of any specific exterior angle = 180° – the size of its adjacent interior angle.



Example 4

Four angles of a pentagon are 100°.

How big is the fifth angle?

The interior angle sum of a pentagon is 3 × 180° = 540°.

Four angles add up to 400°.

The fifth angle must be 540 – 400 = 140°.

EXERCISE 23F

1 Calculate the sum of the interior angles of polygons with these numbers of sides.

a 10 sides **b** 15 sides

c 100 sides **d** 45 sides

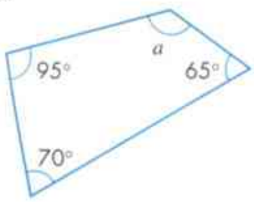
2 Find the number of sides of polygons with these interior angle sums.

a 1260° **b** 2340°

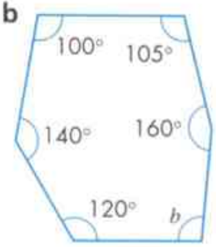
c 18 000° **d** 8640°

3 Calculate the size of the lettered angles in each of these polygons.

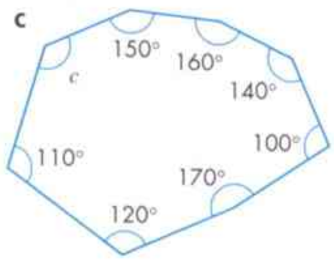
a



b

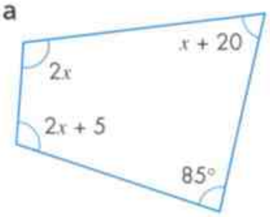


c



4 Find the value of x in each of these polygons.

a



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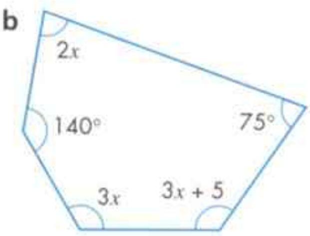

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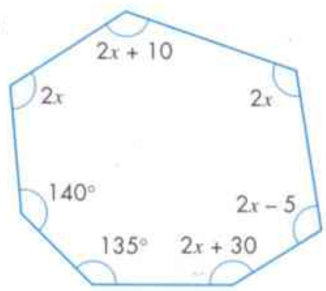

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5 What is the name of the regular polygon in which the interior angles are twice its exterior angles?

6 Wesley measured all the interior angles in a polygon. He added them up to make 991°, but he had missed out one angle.

a What type of polygon did Wesley measure?

b What is the size of the missing angle?

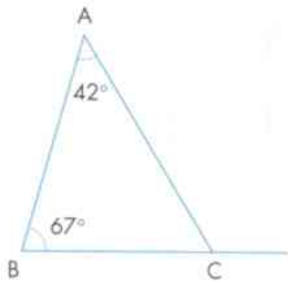
7 **a** In the triangle ABC, angle A is 42° and angle B is 67°.

i Calculate the value of angle C.

ii What is the value of the exterior angle at C?

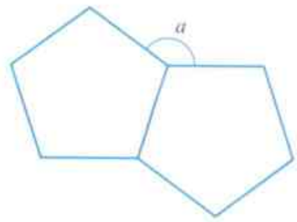
iii What connects the exterior angle at C with the sum of the angles at A and B?

b Prove that any exterior angle of a triangle is equal to the sum of the two opposite interior angles.



8 Two regular pentagons are placed together.

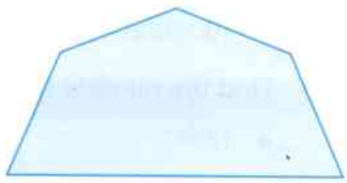
Work out the value of a.



9 A joiner is making tables so that the shape of each one is half a regular octagon, as shown in the diagram.

He needs to know the size of each angle on the top.

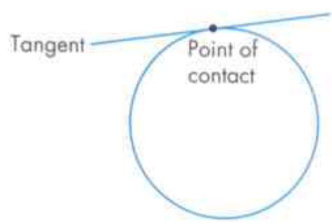
What are the sizes of the angles?



23.7 Tangents and diameters

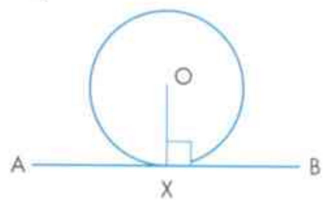
A **tangent** is a straight line that touches a circle at one point only.

This point is called the **point of contact**.



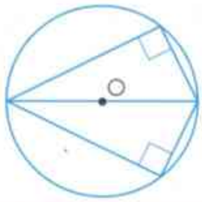
A tangent to a circle is perpendicular to the **radius** drawn to the point of contact.

The radius OX is perpendicular to the tangent AB.



The **diameter** of a circle divides it into two **semi-circles**.

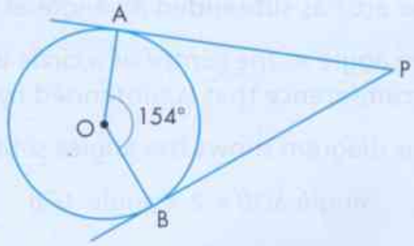
Every angle at the circumference of a semi-circle that is subtended by the diameter of the semi-circle is a right angle.



Example 5

O is the centre of a circle. *AP* and *BP* are tangents.

Calculate the angle at *P*.



The angles at *A* and *B* are right angles.

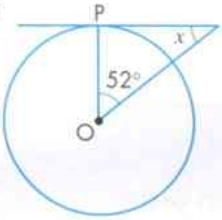
OAPB is a quadrilateral so the interior angles add up to 360°.

Angle *P* is 360 – (90 + 90 + 154) = 26°.

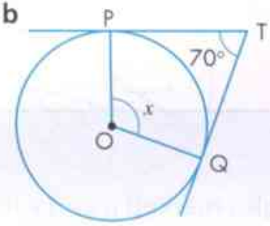
EXERCISE 23G

1 In each diagram, *TP* and *TQ* are tangents to a circle with centre *O*. Find each value of x.

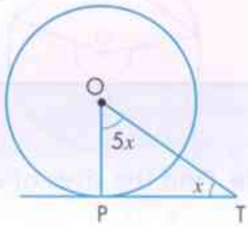
a



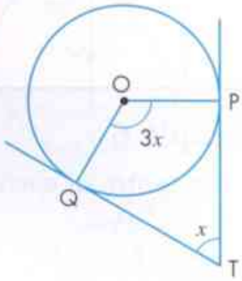
b



c

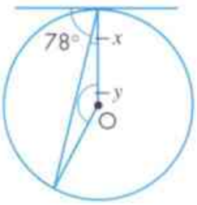


d

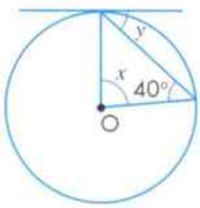


2 Each diagram shows a tangent to a circle with centre *O*. Find the value of x and y in each case.

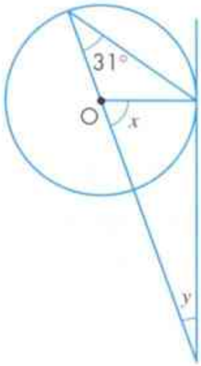
a



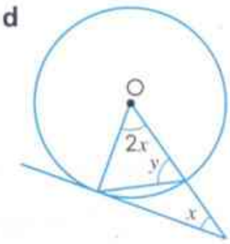
b



c



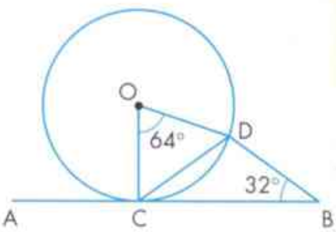
d



3 In the diagram, *O* is the centre of the circle and *AB* is a tangent to the circle at *C*.

Explain why triangle *BCD* is isosceles

Give reasons to justify your answer.



Advice and Tips

Look for isosceles triangles

CORE

23.8 Angles in a circle

E

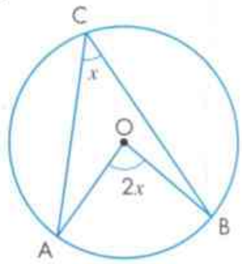
Here are two more theorems you need to know about angles in circles.

If you draw lines from each end of an **arc** to the centre of a circle they form an angle at the centre. The arc has **subtended** an angle at the centre.

The angle at the centre of a circle is twice the angle at the **circumference** that is subtended by the same arc.

This diagram shows the angles subtended by arc AB.

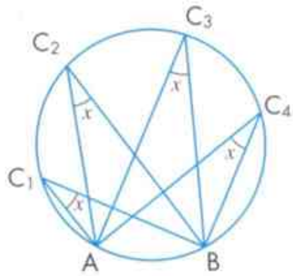
angle $AOB = 2 \times$ angle ACB



Angles subtended at the circumference in the same **segment** of a circle are equal.

Points C_1, C_2, C_3 and C_4 on the circumference are subtended by the same arc AB.

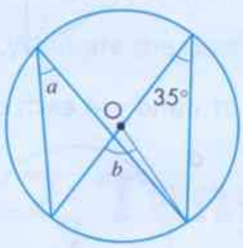
So angle AC_1B = angle AC_2B = angle AC_3B = angle AC_4B



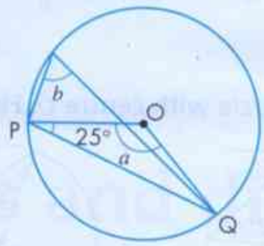
Example 6

O is the centre of each circle. Find the sizes of angles marked a and b in each circle.

a



b



a $a = 35°$ (angles in same segment)

$b = 2 \times 35°$ (angle at centre = twice angle at circumference)

$= 70°$

b With $OP = OQ$, triangle OPQ is isosceles and the sum of the angles in this triangle = 180°

So $a + (2 \times 25°) = 180°$

$a = 180° - (2 \times 25°)$

$= 130°$

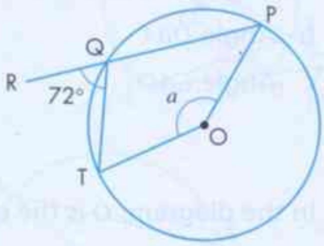
$b = 130° \div 2$ (angle at centre = twice angle at circumference)

$= 65°$

Example 7

O is the centre of the circle. PQR is a straight line.

Find the size of the angle labelled a.



angle PQT = 180° – 72° = 108° (angles on straight line)

The reflex angle angle POT = 2 × 108°
(angle at centre = twice angle at circumference)

= 216°

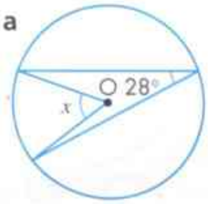
a + 216° = 360° (sum of angles around a point)

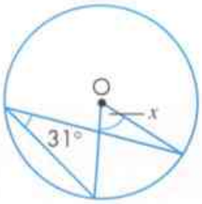
a = 360° – 216°

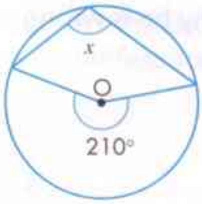
a = 144°

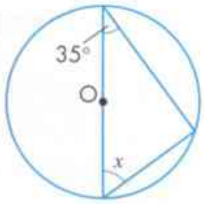
EXERCISE 23H

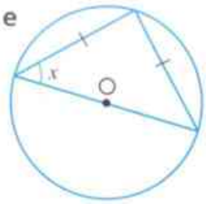
1 Find the value of x in each of these circles with centre O.

a


b


c


d


e


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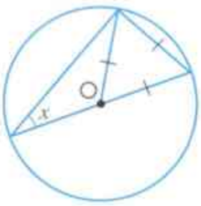
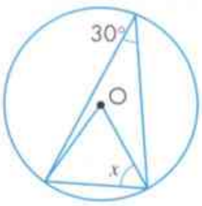
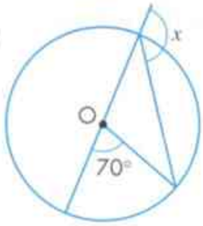
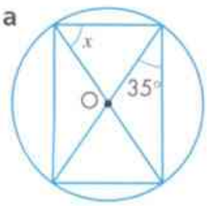


g


h


2 Find the value of x in each of these circles with centre O.

a


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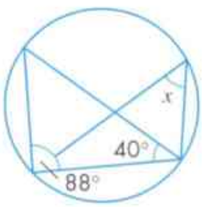

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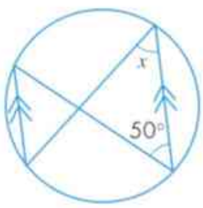

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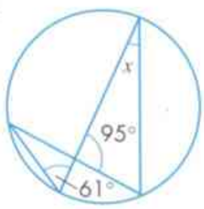

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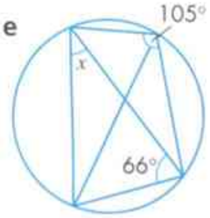

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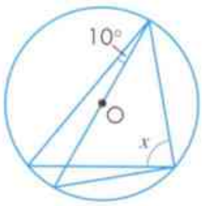

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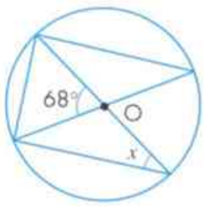

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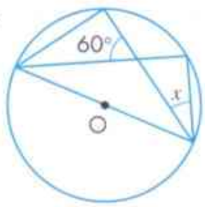

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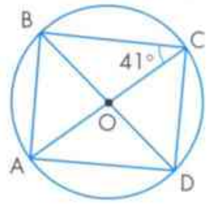
3 In the diagram, O is the centre of the circle. Calculate these angles.

a Angle ADB.

b Angle DBA

c Angle CAD

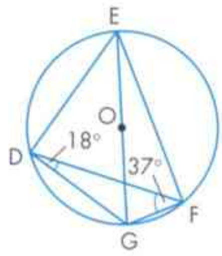


4 In the diagram, O is the centre of the circle. Calculate these angles.

a Angle EDF

b Angle DEG

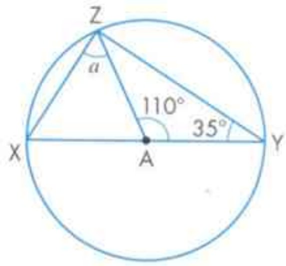
c Angle EGF



5 In the diagram XY is a diameter of the circle and angle AZX is a.

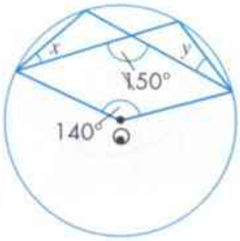
Ben says that the value of a is 50°.

Give reasons to explain why he is wrong.

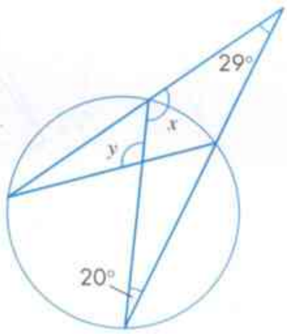


6 Find the values of x and y in each of these circles. O is the centre where shown.

a



b



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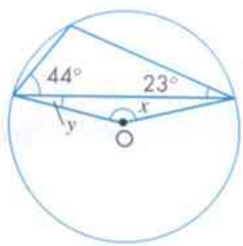
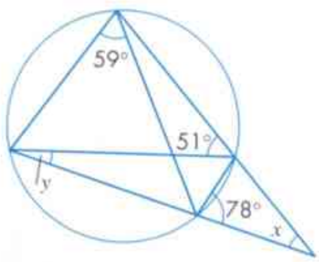
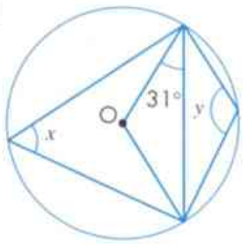


d



e



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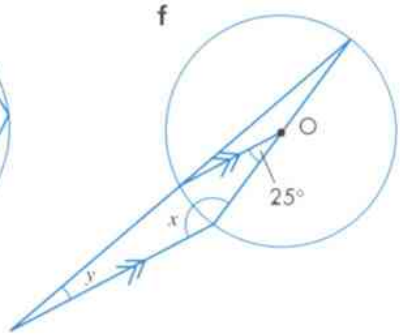
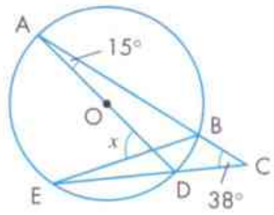


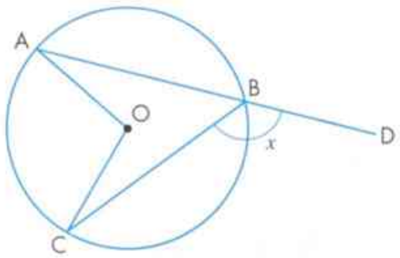
7 In the diagram, O is the centre and AD is the diameter of the circle.

Find the value of x.



8 In the diagram, O is the centre of the circle and angle CBD is x.

Show that the reflex angle AOC is $2x$, giving reasons to explain your answer.

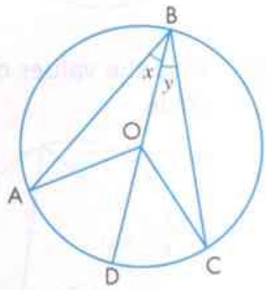


9 A, B, C and D are points on the circumference of a circle with centre O. Angle ABO is $x°$ and angle CBO is $y°$.

a State the value of angle BAO.

b State the value of angle AOD.

c Prove that the angle subtended by the chord AC at the centre of a circle is twice the angle subtended at the circumference.



EXTENDED

23.9 Cyclic quadrilaterals

E

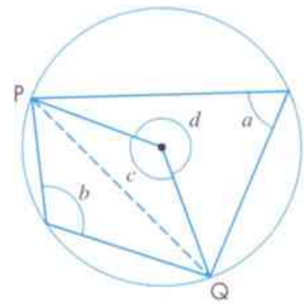
There are *two* segments between points P and Q.

a is the angle in one segment.

b is the angle in the **opposite segment**.

Angles in opposite segments add up to 180°.
These angles (a and b) are **supplementary**.

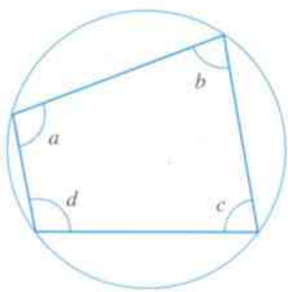
Proof: $a + b = \frac{1}{2}c + \frac{1}{2}d = \frac{1}{2}(c + d) = \frac{1}{2} \times 360 = 180°$



A quadrilateral whose four vertices lie on the circumference of a circle is called a **cyclic quadrilateral**.

The sum of the opposite angles of a cyclic quadrilateral is 180°.

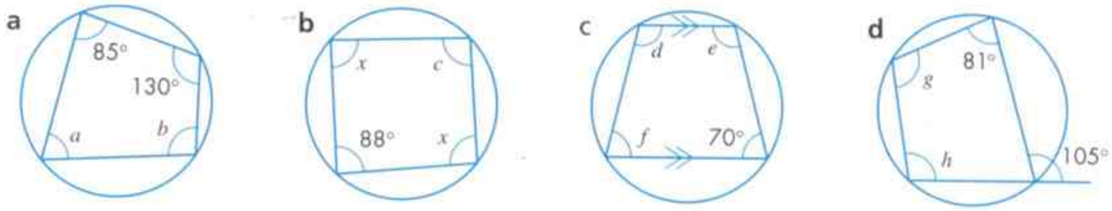
$a + c = 180°$ and $b + d = 180°$

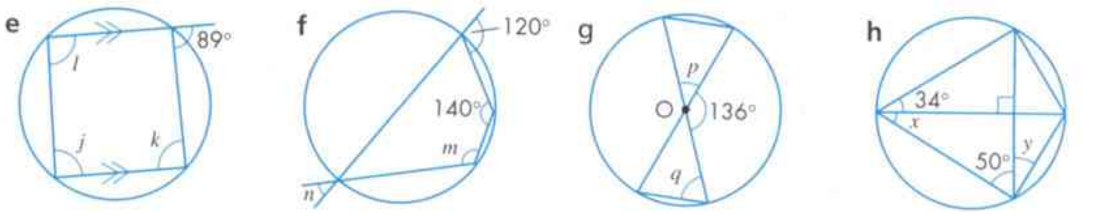


EXERCISE 23I

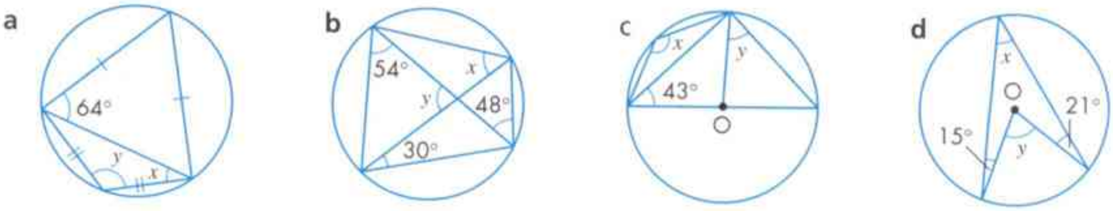
EXTENDED

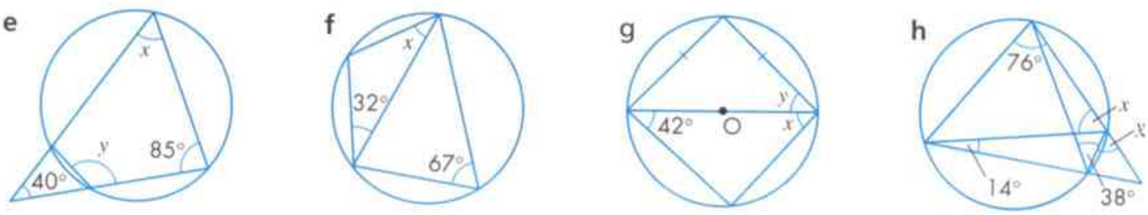
1 Find the sizes of the lettered angles in each of these circles.



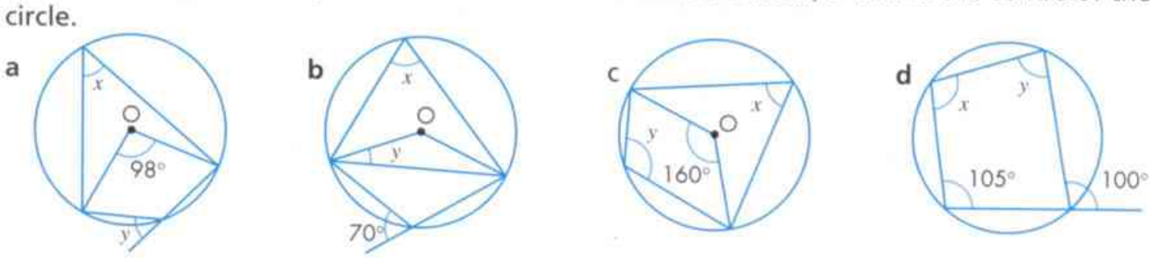


2 Find the values of x and y in each of these circles. Where shown, O marks the centre of the circle.

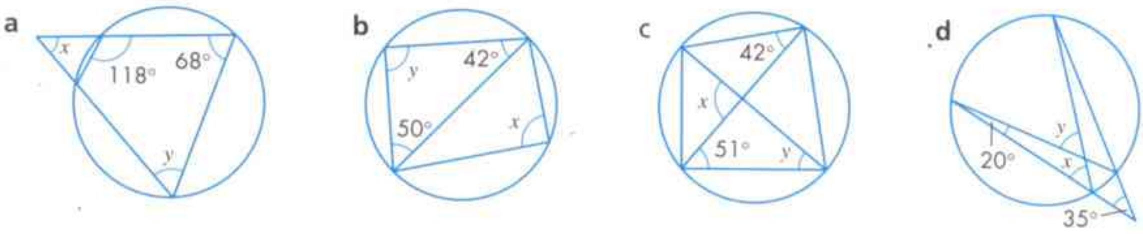




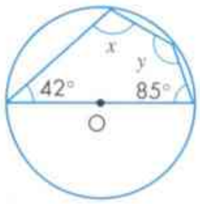
3 Find the values of x and y in each of these circles. Where shown, O marks the centre of the circle.

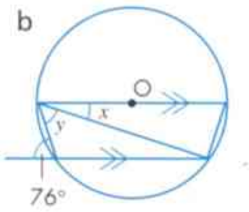


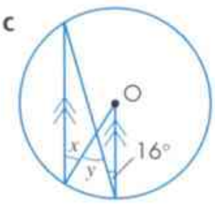
4 Find the values of x and y in each of these circles.

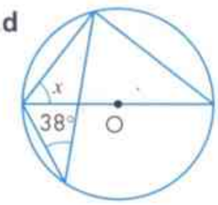


5 Find the values of the angles labelled with letters in each of these circles with centre O.

a


b


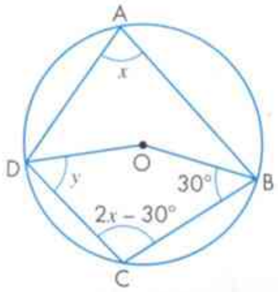
c


d


6 The cyclic quadrilateral $PQRT$ has angle ROQ equal to 38° where O is the centre of the circle. POT is a diameter and parallel to QR. Calculate these angles.

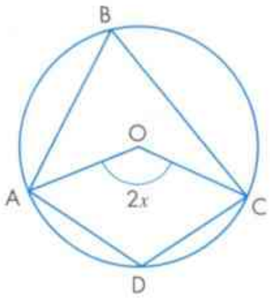
- **a** ROT
- **b** QRT
- **c** QPT

7 In the diagram, O is the centre of the circle.



- **a** Explain why $3x - 30° = 180°$.
- **b** Work out the size of angle CDO, labelled y on the diagram.
  Give reasons in your working.

8 $ABCD$ is a cyclic quadrilateral within a circle centre O and angle AOC is $2x$°.



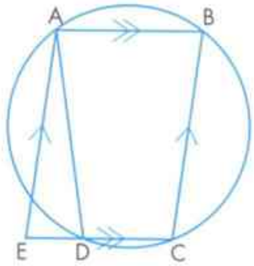
- **a** Write down the size of angle ABC.
- **b** Write down the size of the reflex angle AOC.
- **c** Prove that the sum of a pair of opposite angles of a cyclic quadrilateral is 180°.

EXTENDED

9 In the diagram, $ABCE$ is a parallelogram.

Prove that angle AED = angle ADE.

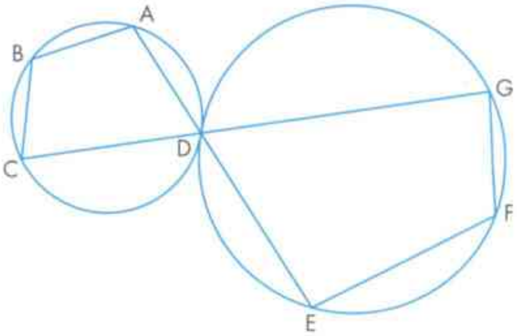
Give reasons in your working.



10 Two circles touch at D.

ADE and CDG are straight lines.

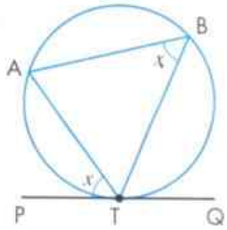
Explain why angles ABC and EFG must be equal in size.



23.10 Alternate segment theorem

E

PTQ is the tangent to a circle at T. The segment containing angle TBA is known as the **alternate segment** of angle PTA, because it is on the other side of the chord AT from angle PTA.

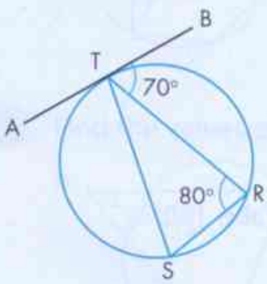


The angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.

angle PTA = angle TBA

Example 8

In the diagram, find a angle ATS and b angle TSR.



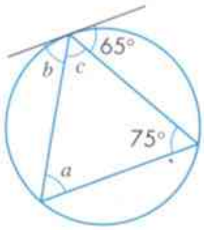
a angle ATS = 80° (angle in alternate segment)

b angle TSR = 70° (angle in alternate segment)

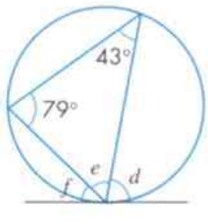
EXERCISE 23J

1 Find the size of each lettered angle.

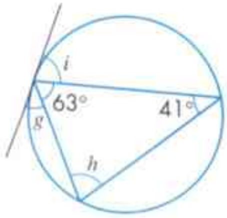
a



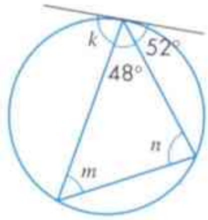
b



c

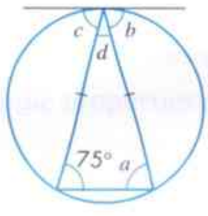


d

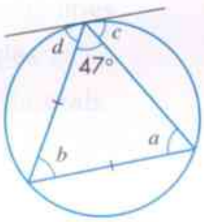


2 In each diagram, find the size of each lettered angle.

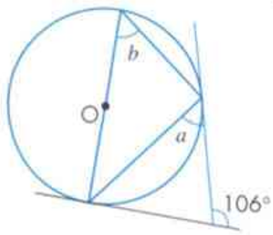
a



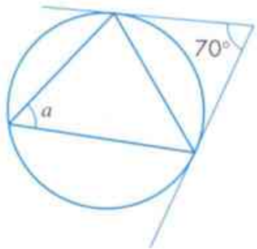
b



c

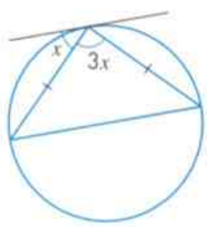


d

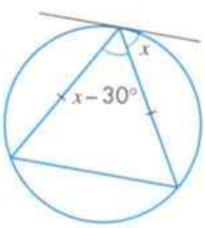


3 In each diagram, find the value of x.

a



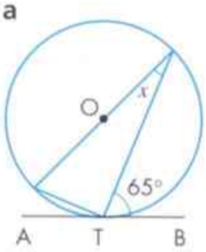
b



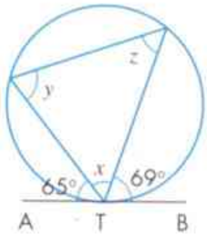
EXTENDED

4 *ATB* is a tangent to each circle with centre *O*. Find the size of each lettered angle.

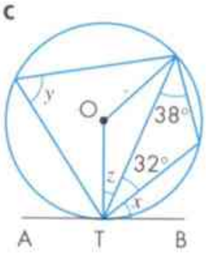
a



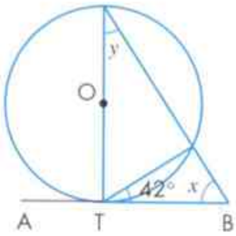
b



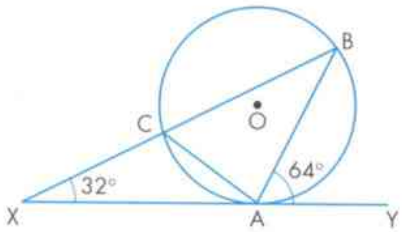
c



d



5 In the diagram, *O* is the centre of the circle.



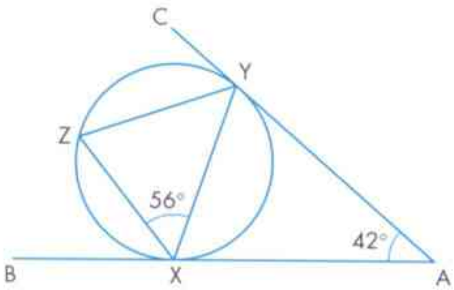
XY is a tangent to the circle at *A*.

BCX is a straight line.

Show that triangle *ACX* is isosceles.

Give reasons to justify your answer.

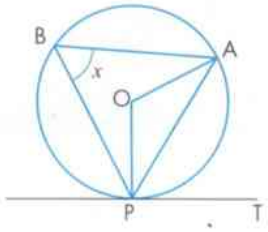
6 *AB* and *AC* are tangents to the circle at *X* and *Y*.



Work out the size of angle *XYZ*.

Give reasons to justify your answer.

7 PT is a tangent to a circle with centre O.
AB are points on the circumference. Angle PBA is $x°$.



a Write down the value of angle AOP in terms of x.

b Calculate the angle OPA in terms of x.

c Prove that the angle APT is equal to the angle PBA.

Check your progress

Core

- I can calculate unknown angles using the following geometrical properties:
  - angles at a point
  - angles on a straight line
  - angles formed within parallel lines
  - angle properties of triangles
  - angle properties of quadrilaterals
  - angle properties of regular polygons
  - an angle in a semi-circle
  - the angle between a tangent and a radius of a circle

Extended

- I can calculate unknown angles using the following geometrical properties:
  - angle properties of irregular polygons
  - angles in the same segment are equal
  - the angle at the centre of a circle is twice the angle at the circumference
  - angle properties of a cyclic quadrilateral
  - the alternate segment theorem

Chapter 24

Geometrical terms and relationships

Topics	Level	Key words
1 Measuring and drawing angles	CORE	right angles, acute angles, obtuse angles, reflex angles, perpendicular, protractor
2 Bearings	CORE	bearing, three-figure bearing
3 Nets	CORE	net, cube, cuboid, prism, pyramid, vertex
4 Congruent shapes	CORE	congruent
5 Congruent triangles	EXTENDED	congruent
6 Similar shapes	CORE	similar, enlargement, linear scale factor, corresponding angles, corresponding sides
7 Areas of similar triangles	EXTENDED	area scale factor
8 Areas and volumes of similar shapes	EXTENDED	solid shapes, volume scale factor

In this chapter you will learn how to:

CORE	EXTENDED
• Use and interpret the geometrical terms: point, line, parallel, bearing, right angle, acute, obtuse and reflex angles, perpendicular, similarity and congruence. (C4.1 and E4.1) • Calculate lengths of similar figures. (C4.4 and E4.4) • Use and interpret vocabulary of triangles, quadrilaterals, circles, polygons and simple solid figures including nets. (C4.1 and E4.1) • Calculate lengths of similar figures. (C4.4 and E4.4) • Interpret and use three-figure bearings measured clockwise from the North (that is 000°–360°). (C6.1 and E6.1) • Recognise congruent shapes. (C4.5)	• Use the relationships between areas of similar triangles, with corresponding results for similar figures. (E4.4) • Extend to volumes and surface areas of similar solids. (E4.4) • Use the basic congruence criteria for triangles (SSS, ASA, SAS, RHS). (E4.5)

Why this chapter matters

Thales of Miletus (624–547 BCE) was a Greek philosopher. Mathematicians believe he was the first person to use similar triangles to find the height of tall objects.

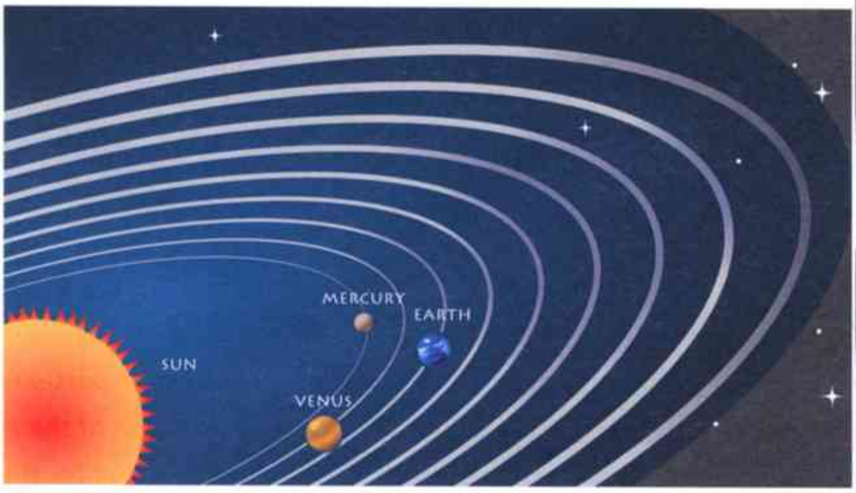
Thales discovered that, at a particular time of day, the height of an object and the length of its shadow were the same. He used this to calculate the height of the Egyptian pyramids.

You can apply the geometry of triangles to calculate heights. You can do this with an instrument called a clinometer. This means you can find the heights of trees, buildings and towers, mountains and other objects which are difficult to measure physically.

Astronomers use the geometry of triangles to measure the distance to nearby stars. They use the Earth's journey in its orbit around the Sun.

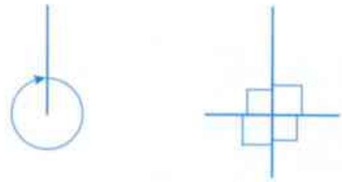
They measure the angle of the star twice, from the same point on Earth, but at opposite ends of its orbit. This gives them angle measurements at a known distance apart and from this triangle they can calculate the distance to the star.

Telescopes and binoculars also use the geometry of triangles.

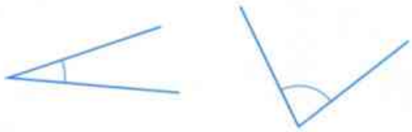


24.1 Measuring and drawing angles

A whole turn is divided into 360° or four **right angles** of 90° each.

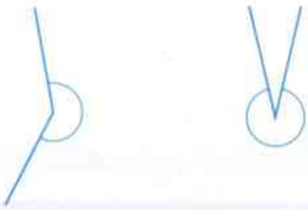


An **acute angle** is less than one right angle.

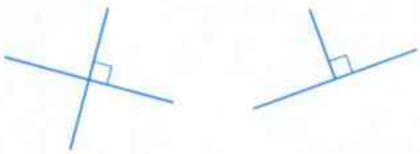


An **obtuse angle** is between one and two right angles (90° and 180°).

A **reflex angle** is between two and four right angles (180° and 360°).



Two lines are **perpendicular** if the angle between them is 90°.



When you are using a **protractor**, it is important that you:

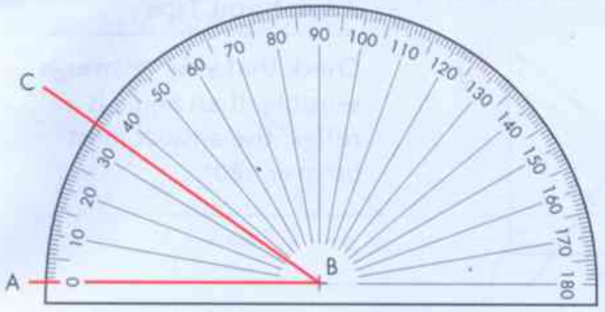
- place the centre of the protractor *exactly* on the corner (vertex) of the angle
- lay the baseline of the protractor *exactly* along one side of the angle.

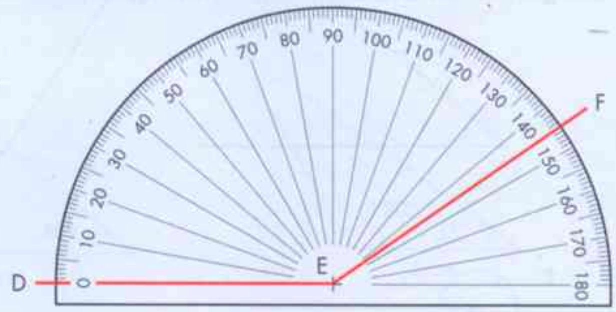
You must follow these two steps to obtain an accurate value for the angle you are measuring.

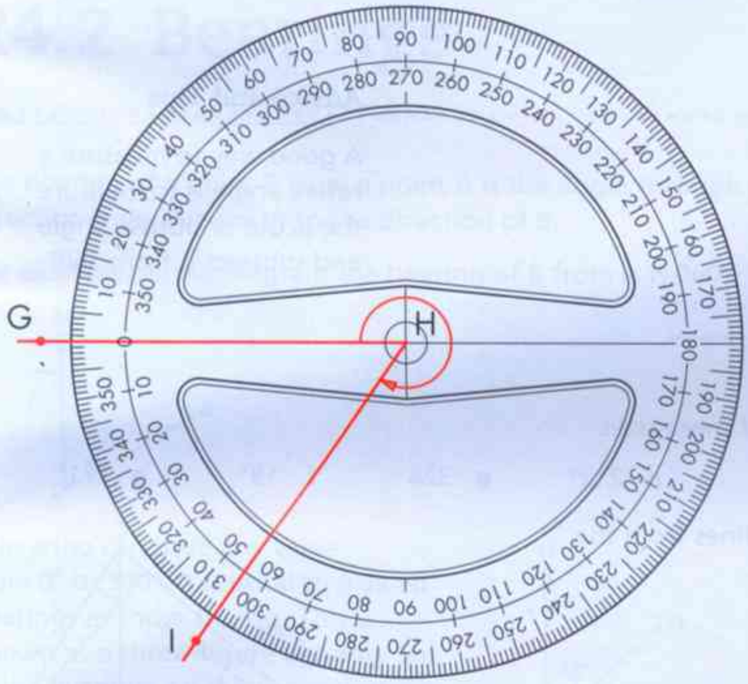
You should already have discovered how easy it is to measure acute angles and obtuse angles, using the common semicircular protractor.

Example 1

Measure the angles ABC, DEF and reflex GHI in the diagrams below.



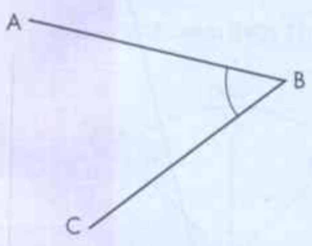




Acute angle ABC is 35° and obtuse angle DEF is 145°.

To measure reflex angles, such as angle GHI, it is easier to use a circular protractor if you have one.

Note the notation for angles.



Angle ABC, or $\angle ABC$, means the angle at B between the lines AB and BC.

Reflex angle GHI is 305°

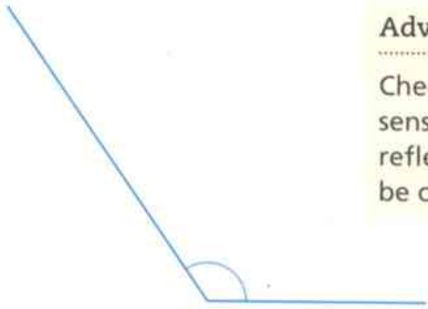
EXERCISE 24A

CORE

1 Use a protractor to measure the size of each marked angle.

a

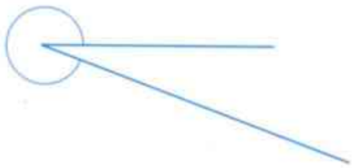
b



Advice and Tips

Check that your answer is sensible. If an angle is reflex, the answer must be over 180°.

c



Advice and Tips

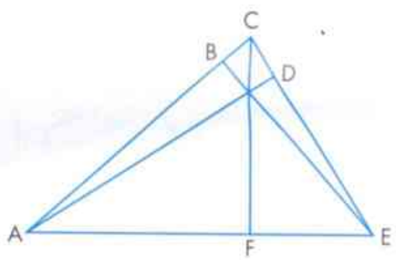
A good way to measure a reflex angle is to measure the acute or obtuse angle and subtract it from 360°.

d

2 Use a protractor to draw angles of these sizes.

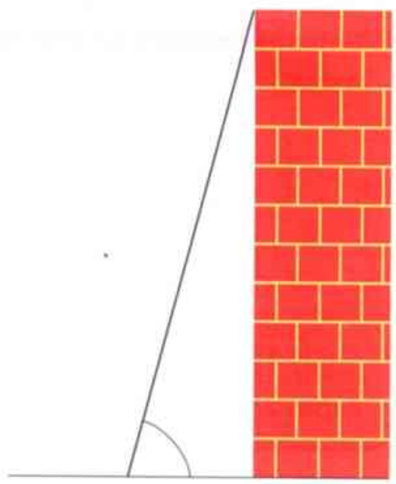
a 30° **b** 125° **c** 90° **d** 212° **e** 324° **f** 19° **g** 171°

3 Find three pairs of perpendicular lines from the following: *AC*, *AD*, *AE*, *BE*, *CE*, *CF*.



4 It is only safe to climb this ladder if the angle between the ground and the ladder is between 72° and 78°.

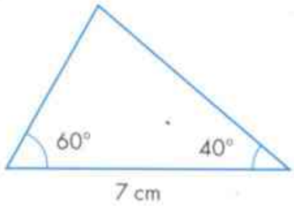
Is it safe for Oliver to climb the ladder?



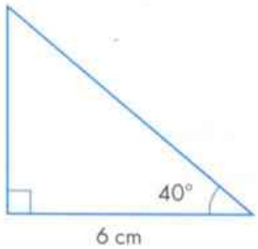
5 An obtuse angle is 10° more than an acute angle.
Write down a possible value for the size of the obtuse angle.

6 Use a ruler and a protractor to draw these triangles accurately. Then measure the unmarked angle in each one.

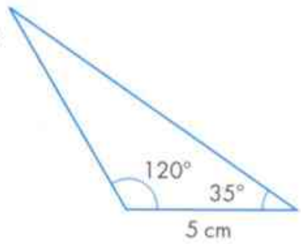
a



b



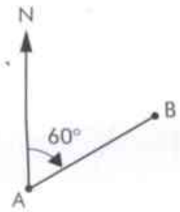
c



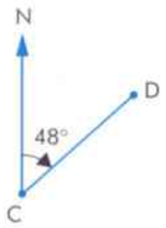
24.2 Bearings

The **bearing** of a point B from a point A is the angle through which you turn *clockwise* as you change direction from *due north* to the direction of B.

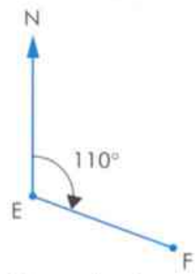
For example, in this diagram the bearing of B from A is 060°.



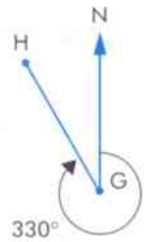
A bearing can have any value from 0° to 360°. It is usual to give all bearings as three figures. This is known as a **three-figure bearing**. So, in the example on the previous page, the bearing is written as 060°, using three figures. Here are three more examples.



D is on a bearing of 048° from C

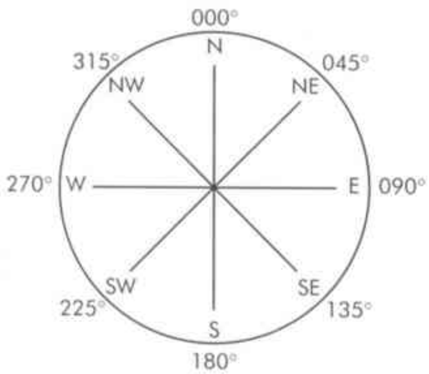


F is on a bearing of 110° from E



H is on a bearing of 330° from G

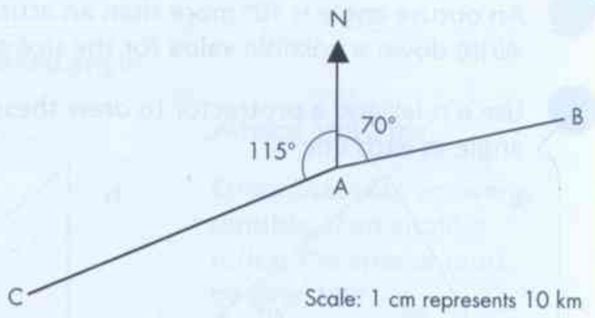
There are eight bearings that you should know. They are shown in the diagram.



Example 2

A, B and C are three towns.

Write down the bearing of B from A and the bearing of C from A.



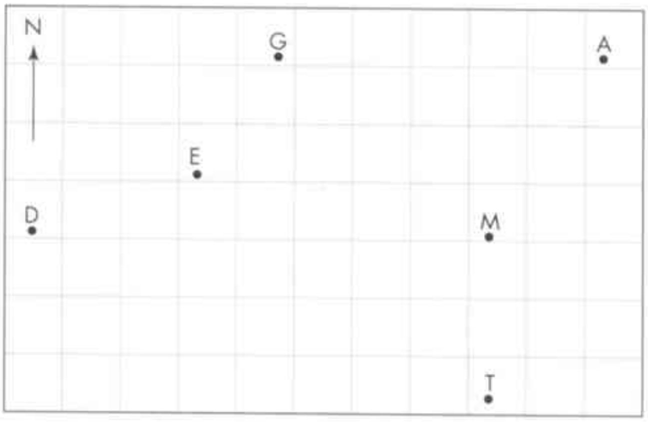
The bearing of B from A is 070°.

The bearing of C from A is 360° − 115° = 245°.

EXERCISE 24B

CORE

1 Look at this map. By measuring angles, find the bearings of:



a T from D

b D from E

c M from D

d G from A

e M from G

f T from M.

2 Draw sketches to illustrate these situations.

a C is on a bearing of 170° from H.

b B is on a bearing of 310° from W.

3 A is due north from C. B is due east from A. B is on a bearing of 045° from C. Sketch the layout of the three points, A, B and C.

4 The Captain decided to sail his ship around the four sides of a square kilometre.

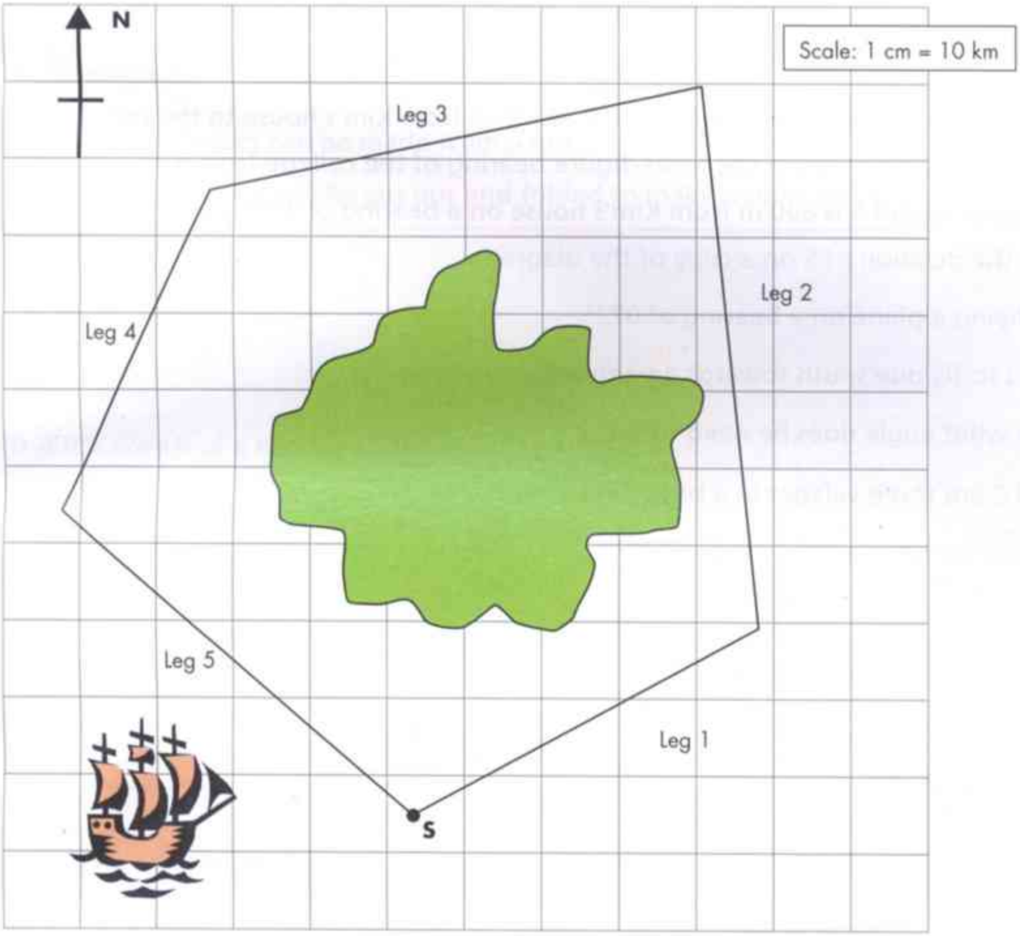
a Assuming he started sailing due north, write down the next three bearings he would use in order to complete the square in a clockwise direction.

b Assuming he started sailing on a bearing of 090°, write down the next three bearings he would use in order to complete the square in an anticlockwise direction.

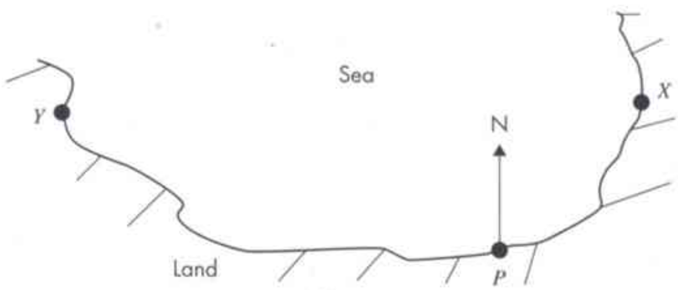
5 The map shows a boat journey around an island, starting and finishing at S. On the map, 1 centimetre represents 10 kilometres. Measure the distance and bearing of each leg of the journey.

Copy and complete this table.

Leg	Actual distance	Bearing
1		
2		
3		
4		
5		



6 The diagram shows a port P and two harbours X and Y on the coast.



a A fishing boat sails to X from P.

What is the three-figure bearing of X from P?

b A yacht sails to Y from P.

What is the three-figure bearing of Y from P?

CORE

7 Draw diagrams to solve these problems.

a The three-figure bearing of A from B is 070°. Work out the three-figure bearing of B from A.

b The three-figure bearing of P from Q is 145°. Work out the three-figure bearing of Q from P.

c The three-figure bearing of X from Y is 324°. Work out the three-figure bearing of Y from X.

8 The diagram shows the position of Kim's house H and the college C.

Scale: 1 cm represents 200 m

a Use the diagram to work out the actual distance from Kim's house to the college.

b Measure and write down the three-figure bearing of the college from Kim's house.

c The supermarket S is 600 m from Kim's house on a bearing of 150°.

Mark the position of S on a copy of the diagram.

9 Chen is flying a plane on a bearing of 072°.

He is told to fly due south towards an airport.

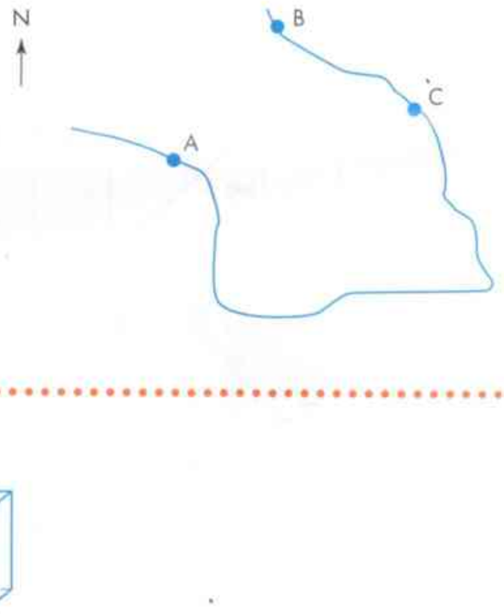
Through what angle does he need to turn?

10 A, B and C are three villages in a bay.

They lie on the vertices of a square.

The bearing of B from A is 030°.

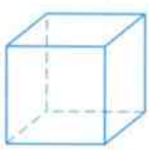
Work out the bearing of A from C.



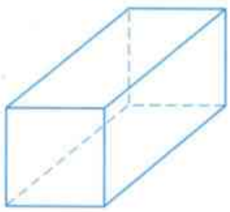
24.3 Nets

You should know these solid shapes:

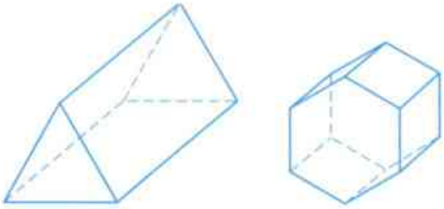
A **cube** has six square faces.



A **cuboid** has rectangular faces.



A **prism** has a uniform cross-section.



A **pyramid** has a polygon-shaped base. The other faces are triangles and meet at the **vertex**.

Each of these solid shapes can be made from a **net**.

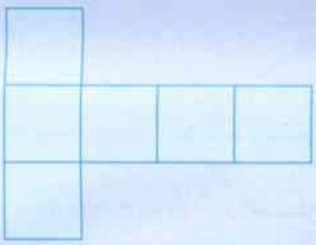
A net is a flat shape which can be cut out and folded to make a solid shape.

Example 3

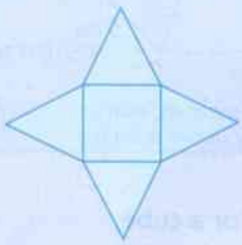
Sketch the net for:

a cube **b** square-based pyramid.

a This is a sketch of a net for a cube.



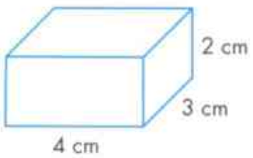
b This is a sketch of a net for a square-based pyramid.



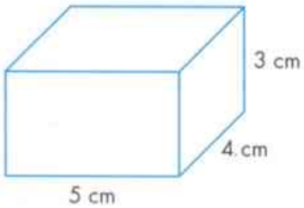
EXERCISE 24C

1 Draw, on squared paper, an accurate net for each of these cuboids.

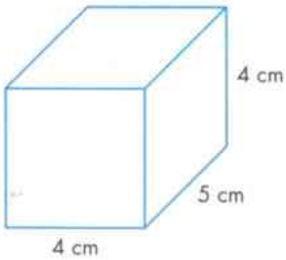
a



b



c



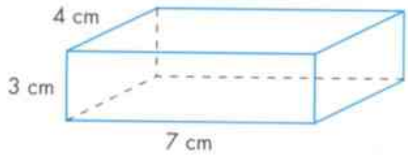
CORE

CORE

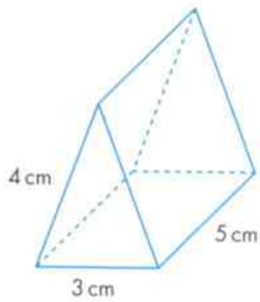
2 Bashira is making an open box from card.

This is a sketch of the box.

Bashira has a piece of card that measures 15 cm by 21 cm. Can she make the box from this card?

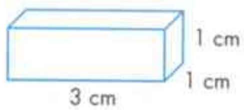


3 The shape on the right is a triangular prism. Its ends are isosceles triangles and its other faces are rectangles. Draw an accurate net for this prism. Use squared paper.



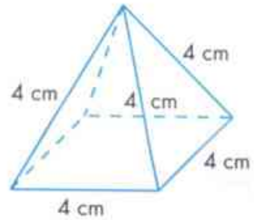
4 Draw the nets of these shapes.

a



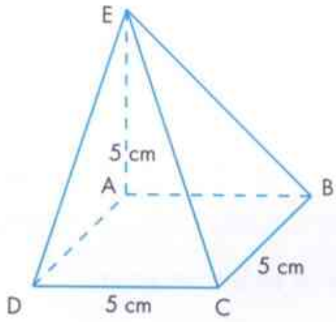
Cuboid

b



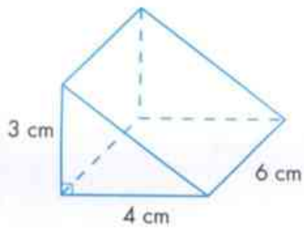
Square-based pyramid

c



Square-based pyramid, with point E directly above point A

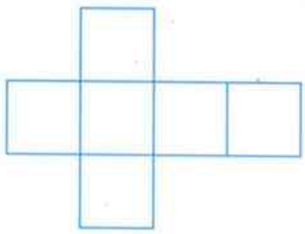
d



Right-angled triangular prism

5 Here is a net for a cube.

How many different nets can you draw for a cube?

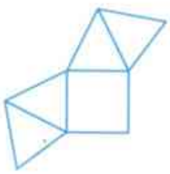


Advice and Tips

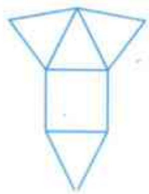
There are 11 altogether. How many can you find?

6 Which of these are nets for a square-based pyramid?

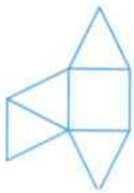
a



b

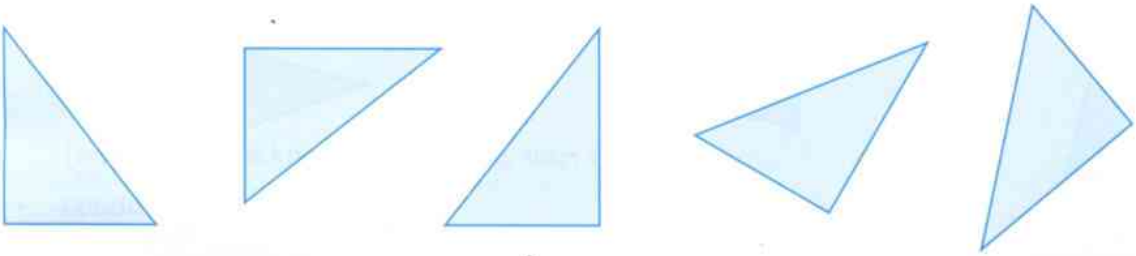


c



24.4 Congruent shapes

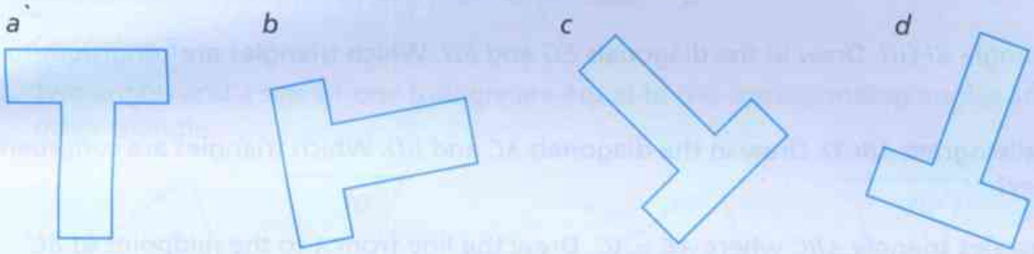
Two-dimensional shapes that are exactly the same size and shape as each other are **congruent**.
For example, although they are in different positions, the triangles below are all congruent, because they are all exactly the same size and shape.



Congruent shapes fit exactly on top of each other. So, one way to see whether shapes are congruent is to trace one of them and check that it covers the other shapes exactly. For some of the shapes, you may have to turn your tracing paper over.

Example 4

Which of these shapes is not congruent to the others?



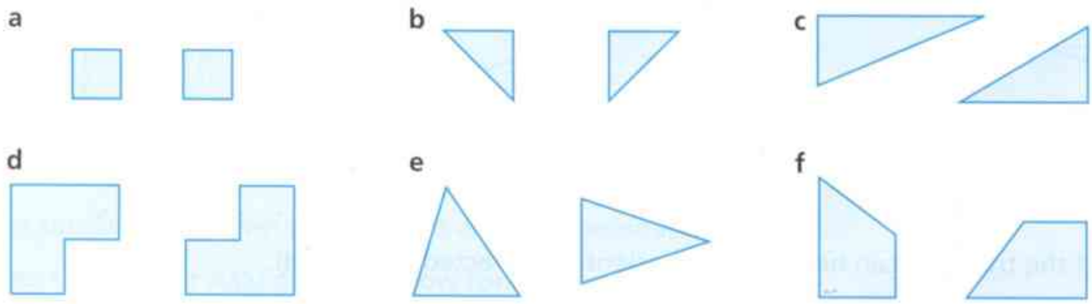
Trace shape **a** and check whether it fits exactly on top of the others.

You should find that shape **b** is not congruent to the others.

EXERCISE 24D

CORE

1 State whether the shapes in each pair, **a** to **f** are congruent or not.

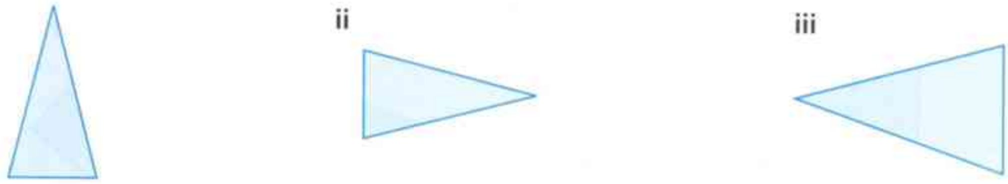


CORE

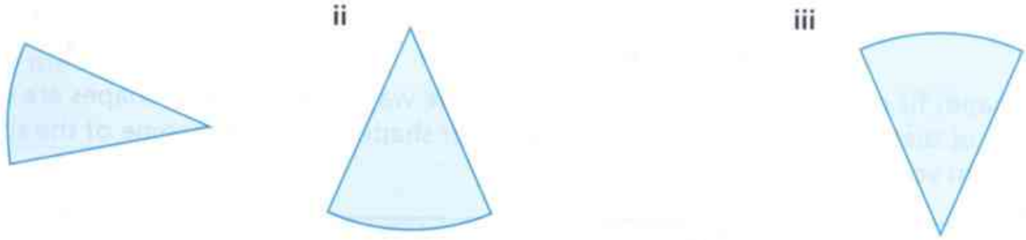
2 Which figure in each group, **a** to **c**, is not congruent to the other two?

a i ii iii

b i ii iii



c i ii iii



3 Draw a square $PQRS$. Draw in the diagonals PR and QS. Which triangles are congruent to each other?

4 Draw a rectangle $EFGH$. Draw in the diagonals EG and FH. Which triangles are congruent to each other?

5 Draw a parallelogram $ABCD$. Draw in the diagonals AC and BD. Which triangles are congruent to each other?

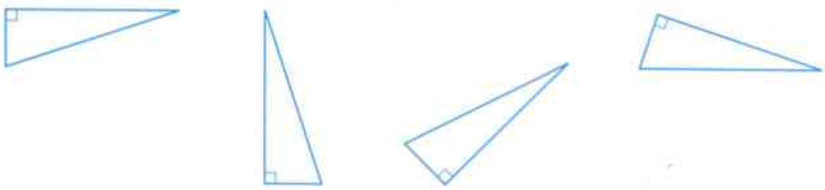
6 Draw an isosceles triangle ABC where $AB = AC$. Draw the line from A to the midpoint of BC. Which triangles are congruent to each other?

24.5 Congruent triangles

E

Two shapes are **congruent** if they are exactly the same size and shape.

For example, these triangles are all congruent.



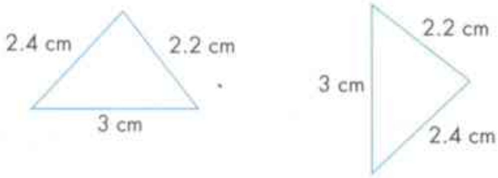
Notice that the triangles can be differently oriented (reflected or rotated).

Conditions for congruent triangles

Any **one** of the following four conditions is sufficient for two triangles to be congruent.

- **Condition 1**

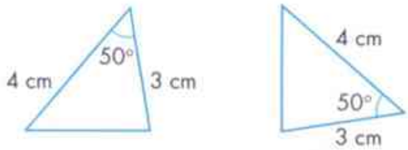
  All three sides of one triangle are equal to the corresponding sides of the other triangle.



  This condition is known as SSS (side, side, side).

- **Condition 2**

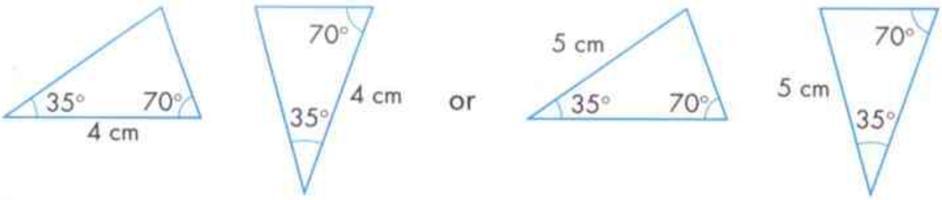
  Two sides and the angle between them of one triangle are equal to the corresponding sides and angle of the other triangle.



  This condition is known as SAS (side, angle, side).

- **Condition 3**

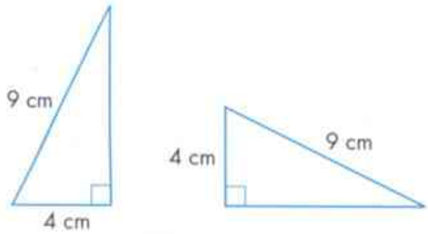
  Two angles and a side of one triangle are equal to the corresponding angles and side of the other triangle.



  This condition is known as ASA (angle, side, angle) or AAS (angle, angle, side).

- **Condition 4**

  Both triangles have a right angle, an equal hypotenuse and another equal side.



  This condition is known as RHS (right angle, hypotenuse, side).

  Note that SSA (or ASS) does not show congruency.

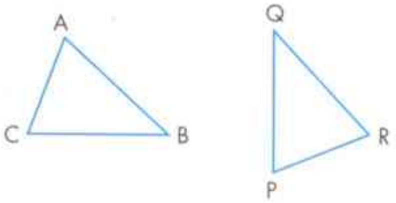
Notation

Once you have shown that triangle ABC is congruent to triangle PQR by one of the above conditions, it means that:

angle A = angle P $AB = PQ$

angle B = angle Q $BC = QR$

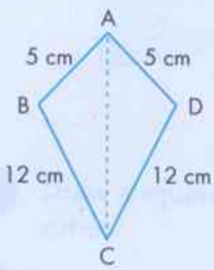
angle C = angle R $AC = PR$



In other words, the points ABC correspond exactly to the points PQR in that order. Triangle ABC is congruent to triangle PQR can be written as $\Delta ABC \equiv \Delta PQR$.

Example 5

$ABCD$ is a kite. Show that triangle ABC is congruent to triangle ADC.



$AB = AD$

$BC = CD$

AC is common

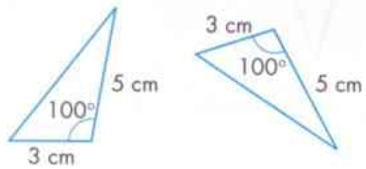
So $\Delta ABC \equiv \Delta ADC$ (SSS)

EXERCISE 24E

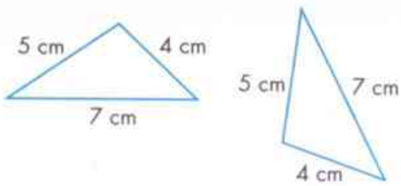
EXTENDED

1 The triangles in each pair are congruent. State the condition that shows that the triangles are congruent.

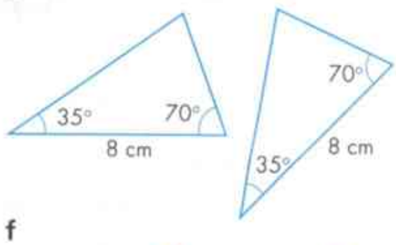
a



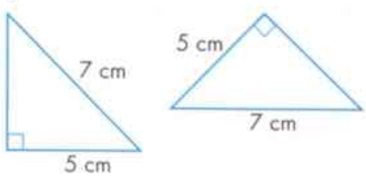
b



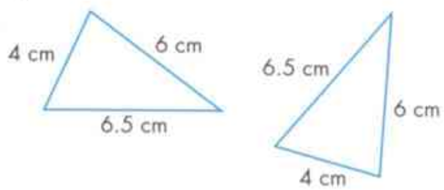
c



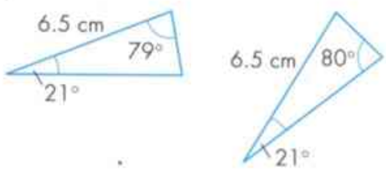
d



e



f



2 The triangles in each pair are congruent. State the condition that shows that the triangles are congruent and say which points correspond to which.

a ABC where AB = 8 cm, BC = 9 cm, AC = 7.4 cm

PQR where PQ = 9 cm, QR = 7.4 cm, PR = 8 cm

b ABC where $AB = 5$ cm, $BC = 6$ cm, angle $B = 35°$
PQR where $PQ = 6$ cm, $QR = 50$ mm, angle $Q = 35°$

3 Triangle ABC is congruent to triangle PQR, angle $A = 60°$, angle $B = 80°$ and $AB = 5$ cm. Find these:

a angle P **b** angle Q **c** angle R **d** PQ

4 $ABCD$ is congruent to $PQRS$, angle $A = 110°$, angle $B = 55°$, angle $C = 85°$ and $RS = 4$ cm. Find these:

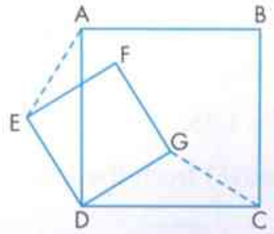
a angle P **b** angle Q **c** angle R **d** angle S **e** CD

5 Draw a rectangle, $EFGH$. Draw in the diagonal EG. Prove that triangle EFG is congruent to triangle EHG.

6 Draw an isosceles triangle (ABC) where $AB = AC$. Draw the line from A to X, the midpoint of BC. Prove that triangle ABX is congruent to triangle ACX.

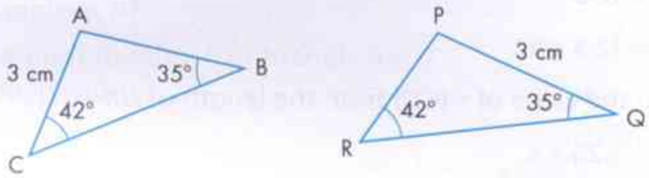
7 In the diagram, $ABCD$ and $DEFG$ are squares.

Use congruent triangles to prove that $AE = CG$.



8 Jez says that these two triangles are congruent because two angles and a side are the same.

Explain why he is wrong.

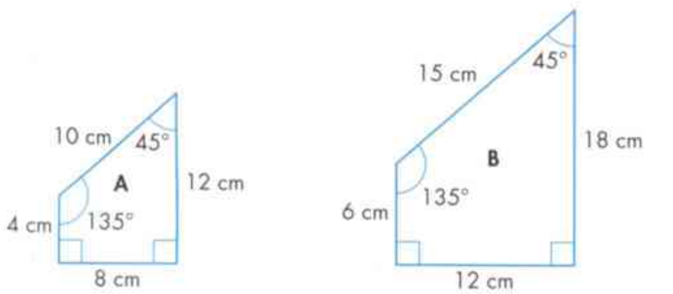


24.6 Similar shapes

Two shapes are **similar** if one is an **enlargement** of the other.

Corresponding angles of similar shapes are equal.

These shapes are similar. Their corresponding angles are equal.



Advice and Tips

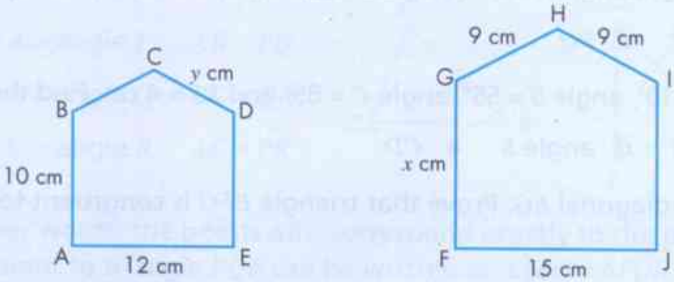
In this section, corresponding angles are those angles that appear in the same position on both shapes. Corresponding sides are those sides that appear in the same position on both shapes.

The ratio of their **corresponding sides** is 2 : 3.

The **linear scale factor** is $\frac{3}{2}$ because the lengths in shape B are $\frac{3}{2}$ of the corresponding lengths in shape A.

Example 6

These two shapes are similar.



Find the values of x and y.

Look for two corresponding sides where you know the lengths.

AE corresponds to FJ and the lengths are 12 cm and 15 cm.

The scale factor is $\frac{15}{12} = 1.25$.

To find the value of x (GF) from the length given for BA:

$x = 1.25 \times AB$

$= 1.25 \times 10$

$= 12.5$

$x = 12.5$ cm

To find the value of y (CD) from the length of HI:

$HI = 1.25 \times y$

$9 = 1.25 \times y$

$y = \frac{9}{1.25}$

$= 7.2$

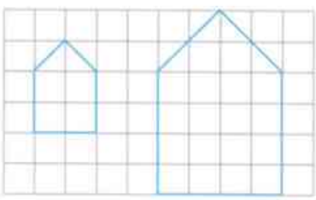
$y = 7.2$ cm

EXERCISE 24F

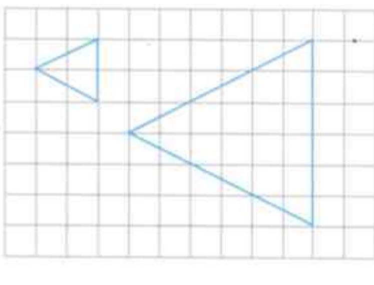
CORE

1 These diagrams are drawn to scale. What is the linear scale factor of the enlargement in each case?

a



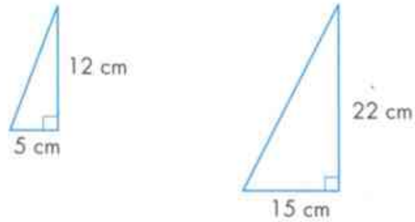
b



2 Are the shapes in each pair similar? If so, give the scale factor. If not, give a reason.

a

b

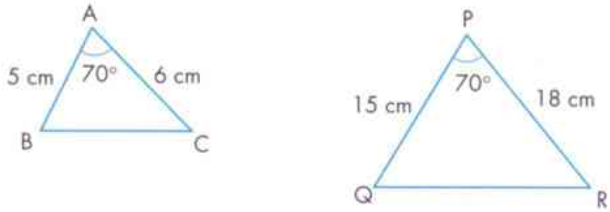


3 **a** Explain why these triangles are similar.

b Give the ratio of the sides.

c Which angle corresponds to angle C?

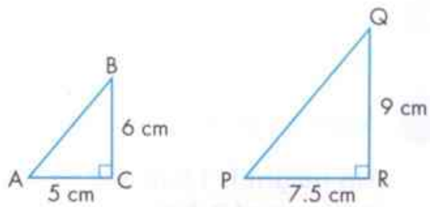
d Which side corresponds to side QP?



4 **a** Explain why these triangles are similar.

b Which angle corresponds to angle A?

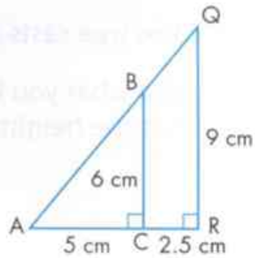
c Which side corresponds to side AC?



5 **a** Explain why triangle ABC is similar to triangle AQR.

b Which angle corresponds to the angle at B?

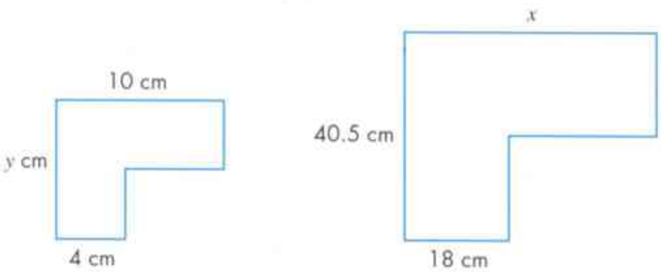
c Which side of triangle AQR corresponds to side AC of triangle ABC? Your answers to question **4** may help you.



6 In the diagrams **a** to **d**, each pair of shapes are similar but not drawn to scale.

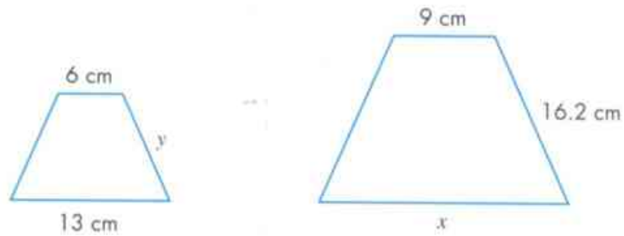
a Find the value of x.

b Find the values of x and y.

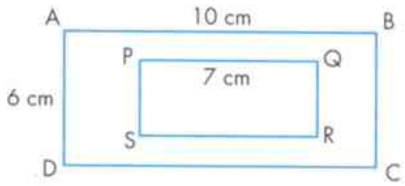


CORE

c Find the values of x and y.



d Calculate the length of QR.



7 a Explain why all squares are similar.

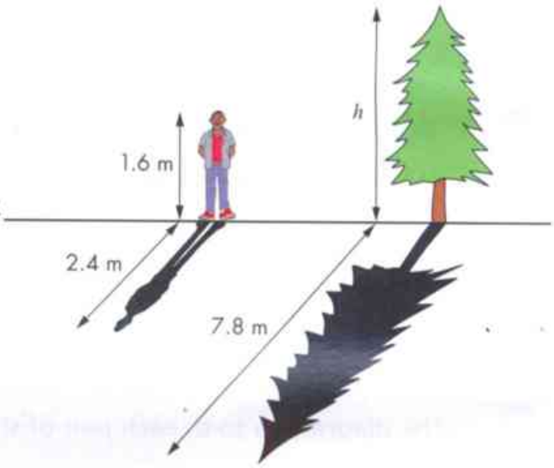
b Are all rectangles similar? Explain your answer.

8 Sean is standing next to a tree.

His height is 1.6 m and he casts a shadow that has a length of 2.4 m.

The tree casts a shadow that has a length of 7.8 m.

Use what you know about similar triangles to work out the height of the tree, h.

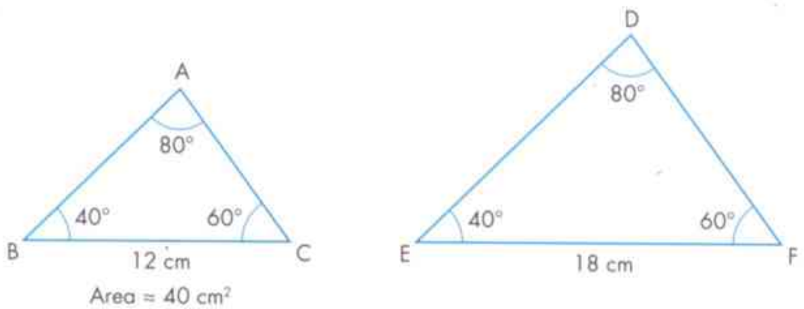


24.7 Areas of similar triangles

E

If two triangles have the same angles then they are similar.

Triangles ABC and DEF are similar.



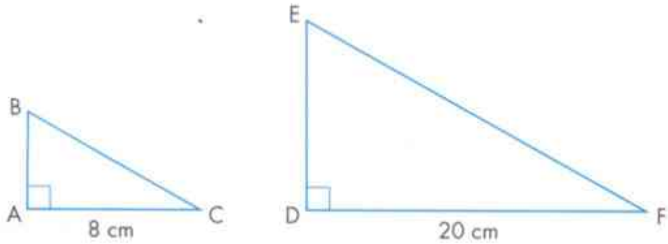
The linear scale factor is $\frac{18}{12} = 1.5$. The **area scale factor** is $1.5^2 = 2.25$.

If the area of triangle ABC is 40 cm^2 then the area of DEF is $40 \times 2.25 = 90$ cm^2.

If the linear scale factor is k, then the area scale factor is k^2.

EXERCISE 24G

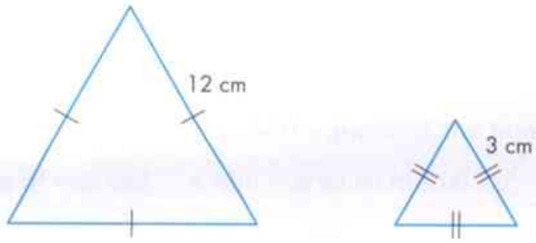
1 These triangles are similar.



a What is the linear scale factor?

b The area of triangle ABC is 20 cm^2.
Calculate the area of triangle DEF.

2 These are equilateral triangles.

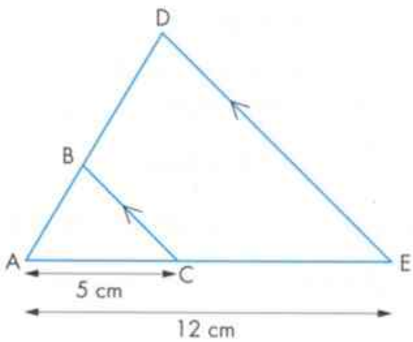


a Explain why they are similar.

b The area of the larger one is 60 cm^2 (to 1 significant figure)
Calculate the area of the smaller one.

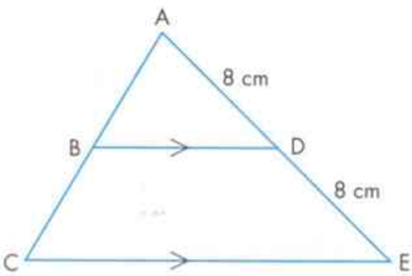
3 The area of triangle ABC is 7 cm^2.

Calculate the area of triangle ADE.



4 The area of triangle ABD is 25 cm^2.

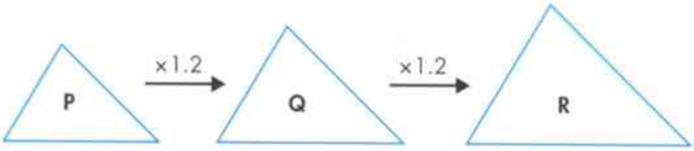
Calculate the area of the trapezium $CBDE$.



5 P is enlarged by a linear scale factor of 1.2 to make Q.

Q is enlarged by a linear scale factor of 1.2 to make R.

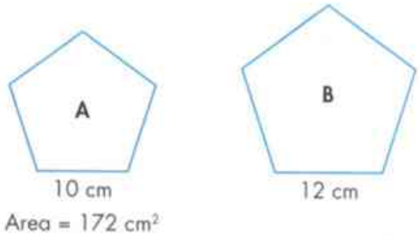
The area of Q is 100 cm^2.



a What is the area of R?

b What is the area of P?

6 The area scale factor also applies to other similar shapes.

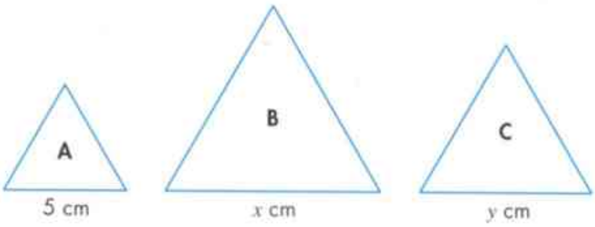


A and B are regular pentagons.

a Explain why they are similar.

b Calculate the area of B.

7



A, B and C are equilateral triangles.

B is four times the area of A.

a What is the linear scale factor?

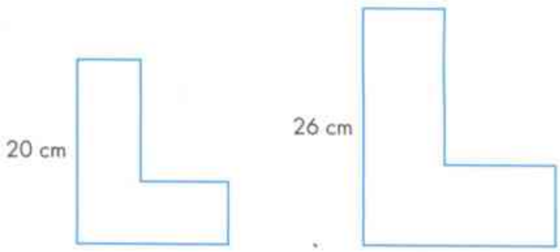
b What is the value of x?

c C is twice the area of A.
Calculate the value of y.

8 These shapes are similar.

The smaller one has an area of 210 cm^2.

Find the area of the larger one.



9 A photocopier has a setting for enlargements with a scale factor of 1.41%.

What is special about this particular value?

10 All circles are similar.

If a circle with a diameter of 8 cm has an area of 50.3 cm^2, what is the area of a circle with diameter 6 cm?

24.8 Areas and volumes of similar shapes

E

You saw that if two shapes are similar and the linear scale factor is k then the area scale factor is k^2.

Two **solid shapes** are similar if corresponding lengths are in the same ratio and corresponding angles are equal. In that case the **volume scale factor** is k^3.

Generally, the relationship between similar shapes can be expressed as:

Length ratio $x : y$ Area ratio $x^2 : y^2$ Volume ratio $x^3 : y^3$

Example 7

A model yacht is made to a scale of $\frac{1}{20}$ of the size of the real yacht. The area of the sail of the model is 150 cm². What is the area of the sail of the real yacht?

At first, it may appear that you do not have enough information to solve this problem, but you can do it like this.

Linear scale factor = 1 : 20

Area scale factor = 1 : 400 (square of the linear scale factor)

Area of real sail = 400 × area of model sail

= 400 × 150 cm²

= 60 000 cm² = 6 m²

Example 8

A bottle has a base radius of 4 cm, a height of 15 cm and a capacity of 650 cm³. A similar bottle has a base radius of 3 cm.

a What is the length ratio?

b What is the volume ratio?

c What is the volume of the smaller bottle?

a The length ratio is given by the ratio of the two radii, that is 4 : 3.

b The volume ratio is therefore $4^3 : 3^3 = 64 : 27$.

c Let v be the volume of the smaller bottle. Then the volume ratio is:

$$\frac{\text{volume of smaller bottle}}{\text{volume of larger bottle}} = \frac{v}{650} = \frac{27}{64}$$

$$\Rightarrow v = \frac{27 \times 650}{64} = 274 \text{ cm}^3 \text{ (3 significant figures)}$$

EXTENDED

EXERCISE 24H

1 The length ratio between two similar solids is 2 : 5.

a What is the area ratio between the solids?

b What is the volume ratio between the solids?

2 The length ratio between two similar solids is 4 : 7.

a What is the area ratio between the solids?

b What is the volume ratio between the solids?

3 Copy and complete this table.

Linear scale factor	Linear ratio	Linear fraction	Area scale factor	Volume scale factor
2	1 : 2	$\frac{2}{1}$		
3				
$\frac{1}{4}$	4 : 1	$\frac{1}{4}$		$\frac{1}{64}$
			25	
				$\frac{1}{1000}$

4 A shape has an area of 15 cm^2. What is the area of a similar shape with lengths that are three times the corresponding lengths of the first shape?

5 A toy brick has a surface area of 14 cm^2. What would be the surface area of a similar toy brick with lengths that are:

a twice the corresponding lengths of the first brick

b three times the corresponding lengths of the first brick?

6 A rug has an area of 12 m^2. What area would be covered by rugs with lengths that are:

a twice the corresponding lengths of the first rug

b half the corresponding lengths of the first rug?

7 A brick has a volume of 300 cm^3. What would be the volume of a similar brick whose lengths are:

a twice the corresponding lengths of the first brick

b three times the corresponding lengths of the first brick?

8 A tin of paint, 6 cm high, holds a half a litre of paint. How much paint would go into a similar tin which is 12 cm high?

9 A model statue is 10 cm high and has a volume of 100 cm^3. The real statue is 2.4 m high. What is the volume of the real statue? Give your answer in m^3.

10 A small tin of paint costs \$0.75. What is the cost of a larger similar tin with height twice that of the smaller tin? Assume that the cost is based only on the volume of paint in the tin.

11 A small box of width 2 cm has a volume of 10 cm^3. What is the width of a similar box with a volume of 80 cm^3?

12 A cinema sells popcorn in two different-sized tubs that are similar in shape.

Show that it is true that the big tub is better value.

13 The diameters of two ball bearings are given below.

Work out:

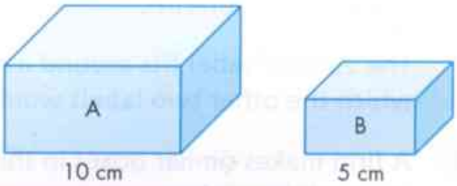
a the ratio of their radii

b the ratio of their surface areas

c the ratio of their volumes.

14 Cuboid A is similar to cuboid B.

The length of cuboid A is 10 cm and the length of cuboid B is 5 cm.



The volume of cuboid A is 720 cm^3.

Zainab says that the volume of cuboid B must be 360 cm^3.

Explain why she is wrong.

More complex problems on area and volume ratios

In some problems involving similar shapes, the length ratio is not given, so you need to start with the area ratio or the volume ratio. Then you will need to find the length ratio in order to proceed with the solution.

Example 9

A manufacturer makes a range of clown hats that are all similar in shape. The smallest hat is 8 cm tall and uses 180 cm^2 of card. What will be the height of a hat made from 300 cm^2 of card?

The area ratio is 180 : 300

Therefore, the length ratio is $\sqrt{180} : \sqrt{300}$ (Do not calculate these yet.)

Let the height of the larger hat be H, then:

$$\frac{H}{8} = \frac{\sqrt{300}}{\sqrt{180}} = \sqrt{\frac{300}{180}}$$

$$\Rightarrow H = 8 \times \sqrt{\frac{300}{180}} = 10.3 \text{ cm (1 decimal place)}$$

Example 10

Two similar tins hold respectively 1.5 litres and 2.5 litres of paint. The area of the label on the smaller tin is 85 cm². What is the area of the label on the larger tin?

The volume ratio is 1.5 : 2.5

Therefore, the length ratio is $\sqrt[3]{1.5} : \sqrt[3]{2.5}$ (Do not calculate these yet.)

So the area ratio is $(\sqrt[3]{1.5})^2 : (\sqrt[3]{2.5})^2$

Let the area of the label on the larger tin be A, then:

$$\frac{A}{85} = \frac{(\sqrt[3]{2.5})^2}{(\sqrt[3]{1.5})^2} = \left(\sqrt[3]{\frac{2.5}{1.5}}\right)^2 \quad \Rightarrow A = 85 \times \left(\sqrt[3]{\frac{2.5}{1.5}}\right)^2 = 119 \text{ cm}^2 \text{ (3 significant figures)}$$

EXERCISE 24I

EXTENDED

1 A firm produces three sizes of similar-shaped labels for its products. Their areas are 150 cm², 250 cm² and 400 cm².

The 250 cm² label fits around a can of height 8 cm. Find the heights of similar cans around which the other two labels would fit.

2 A firm makes similar boxes in three different sizes: small, medium and large. These are the areas of their lids.

Small: 30 cm² Medium: 50 cm² Large: 75 cm²

The medium box is 5.5 cm high. Find the heights of the other two sizes.

3 A cone of height 8 cm can be made from a piece of card with an area of 140 cm². What is the height of a similar cone made from a similar piece of card with an area of 200 cm²?

4 It takes 5.6 litres of paint to paint a chimney that is 3 m high. What is the tallest similar chimney that can be painted with 8 litres of paint?

5 A piece of card, 1200 cm² in area, will make a tube 13 cm long. What is the length of a similar tube made from a similar piece of card with an area of 500 cm²?

6 If a television screen of area 220 cm² has a diagonal length of 21 cm, what is the diagonal length of a similar screen of area 350 cm²?

7 There are two similar bronze statues. One has a mass of 300 g, the other has a mass of 2 kg. The height of the smaller statue is 9 cm.

What is the height of the larger statue?

8 These are the sizes of the labels around three similar cans.

Small can: 24 cm² Medium can: 46 cm² Large can: 78 cm²

The medium size can is 6 cm tall with a mass of 380 g. Calculate:

a the heights of the other two sizes b the masses of the other two sizes.

9 A statue has a mass of 840 kg. A similar statue was made out of the same material but is only two-fifths the height of the first one. What was the mass of the smaller statue?

10 A wooden model stands on a base of area 12 cm^2. A similar wooden model stands on a base of area 7.5 cm^2.

Calculate the mass of the smaller model if the larger one has a mass of 3.5 kg.

11 Stefan fills two similar jugs with orange juice.

The first jug holds 1.5 litres of juice and has a base diameter of 8 cm.

The second jug holds 2 litres of juice. Work out the base diameter of the second jug.

12 The total surface areas of two similar cuboids are 500 cm^2 and 800 cm^2.

If the width of one of the cuboids is 10 cm, calculate the two possible widths for the other cuboid.

13 The volumes of two similar cylinders are 256 cm^3 and 864 cm^3.

Which of these is the ratio of their surface areas?

a 2 : 3 **b** 4 : 9 **c** 8 : 27

Check your progress

Core

- I can measure and draw angles
- I can understand and use geometrical terms including right angle, acute angle, obtuse angle, reflex angle and perpendicular lines
- I can understand and use the terms similar and congruent
- I know and can use the names of polygons and solid figures including nets
- I can calculate lengths in similar figures
- I can interpret and use three-figure bearings

Extended

- I can use the relationship between the area of similar triangles and similar figures
- I can use the relationship between the volumes of similar figures
- I can use the relationship between the surface areas of similar figures
- I can use the congruence criteria for triangles

Chapter 25

Geometrical constructions

Topics	Level	Key words
1 Constructing shapes	CORE	construct, ruler, protractor, compasses, set square
2 Scale drawings	CORE	scale drawing

In this chapter you will learn how to:

CORE

- Measure and draw lines and angles. (C4.2 and E4.2)
- Construct other simple geometrical figures from given data using protractors and set squares as necessary. (C4.2 and E4.2)
- Read and make scale drawings. (C4.3 and E4.3)

Why this chapter matters

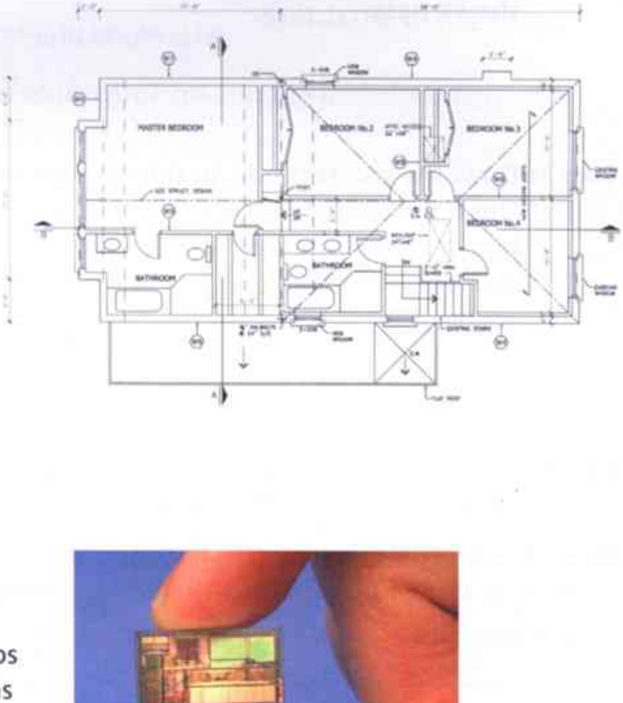
Engineers, town planners, architects, surveyors, builders and computer designers all need to work with great precision. Some projects involve working with very big lengths or distances; others involve very tiny ones. So how do they manage to work with these difficult measures?

The answer is that they all work with drawings drawn to scale. This allows them to represent lengths they cannot easily measure with standard equipment.

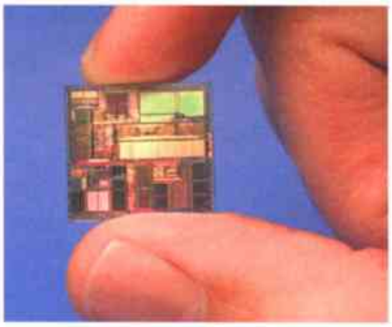
In a scale drawing, one length is used to represent another.

For example, a map cannot be drawn to the same size as the area it represents. The measurements are scaled down, to make a map of a size that can be conveniently used by drivers, tourists and walkers.

Architects use scale drawings to show views of a plannedhouse from different directions.



In computer design, people who design microchips need to scale up their drawings, as the dimensions they work with are so small. Use the internet to research more about how scale drawings are used.



25.1 Constructing shapes

You need to be able to draw a triangle when you are given the lengths of its sides.

You need a ruler and a pair of compasses to do this.

Use a sharp pencil and do not use a pen.

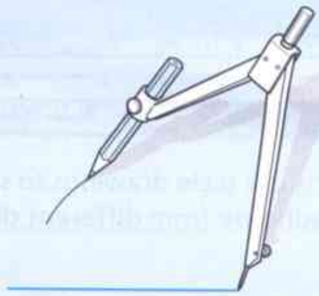
Leave any construction lines on your drawing. Do not erase them.

Example 1

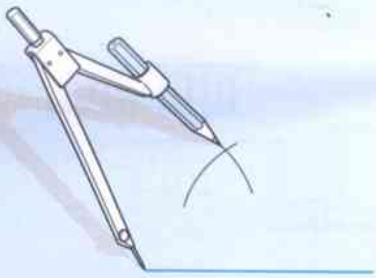
Construct a triangle with sides that are 5 cm, 4 cm and 6 cm long.

Step 1: Draw the longest side as the base. In this case, the base will be 6 cm, which you draw along a ruler. (The diagrams in this example are drawn at half-size.)

Step 2: Draw the second longest side, in this case the 5 cm side. Open the compasses to a radius of 5 cm (the length of the side), place the point on one end of the 6 cm line and draw a short faint arc, as shown here.

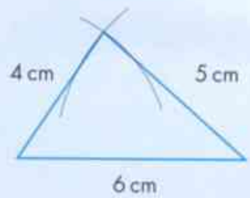


Step 3: Draw the shortest side, in this case the 4 cm side. Open the compasses to a radius of 4 cm, place the point on the other end of the 6 cm line and draw a second short faint arc to intersect the first arc, as shown here.



Step 4: Complete the triangle by joining each end of the base line to the point where the two arcs intersect.

Note: The arcs are construction lines and so you must leave them in, to show how you constructed the triangle.

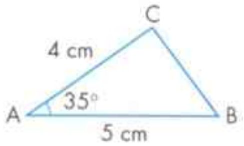


EXERCISE 25A

1 Draw each triangle accurately and measure the sides and angles not given in the diagram.

> **Advice and Tips**
>
> Always make a sketch if one is not given in the question.

a



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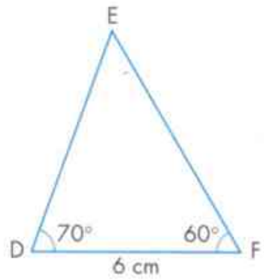

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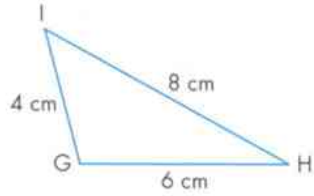

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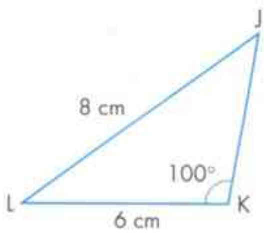

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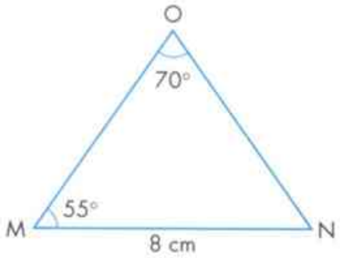

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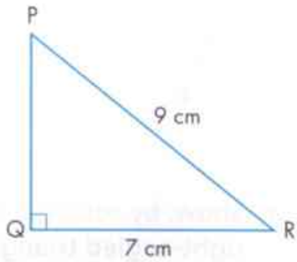


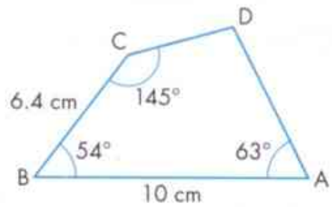
2 **a** Draw a triangle ABC, with $AB = 7$ cm, $BC = 6$ cm and $AC = 5$ cm.

b Measure the sizes of angle ABC, angle BCA and angle CAB.

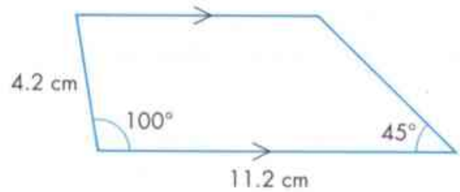
> **Advice and Tips**
>
> Sketch the triangle first.

3 Draw an isosceles triangle that has two sides of length 7 cm and the included angle of 50°. Measure the length of the base of the triangle.

4 Make an accurate drawing of this quadrilateral.

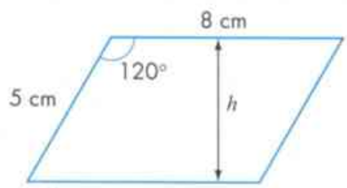


5 Make an accurate drawing of this trapezium.



6 Construct an equilateral triangle of side length 5 cm. Measure the height of the triangle.

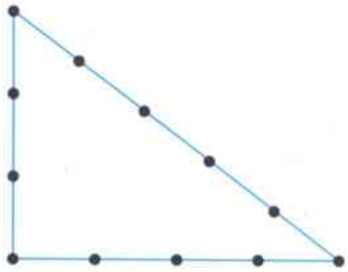
7 Construct a parallelogram with sides of length 5 cm and 8 cm and with an angle of 120° between them. Measure the height of the parallelogram.



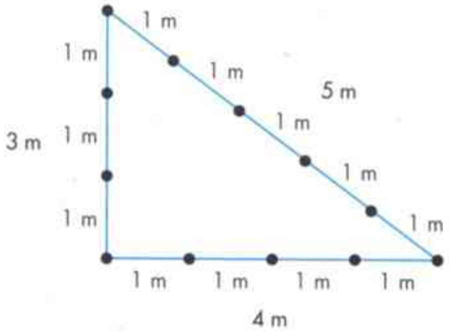
CORE

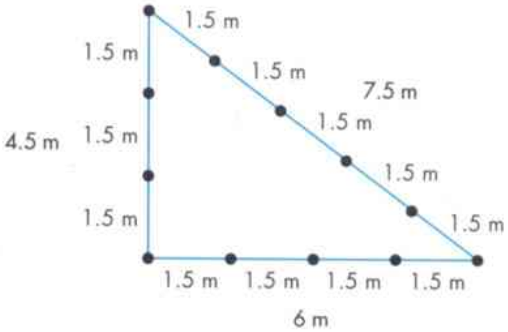
8 A rope has 12 equally-spaced knots. It can be laid out to give a triangle, like this.

It will always be a right-angled triangle.



Here are two examples of such ropes.





a Show, by constructing both of the above triangles (use a scale of 1 cm : 1 m), that each is a right-angled triangle.

b Choose a different triangle that you think might also be right-angled. Use the same knotted-rope idea to check.

9 Construct the triangle with the largest area that has a total perimeter of 12 cm.

10 Anil says that, as long as he knows all three angles of a triangle, he can draw it. Explain why Anil is wrong.

25.2 Scale drawings

A **scale drawing** is an accurate representation of a real object.

Scale drawings are usually smaller in size than the original objects. However, in certain cases, they are enlargements. Examples of these are drawings of miniature electronic circuits and very small watch movements.

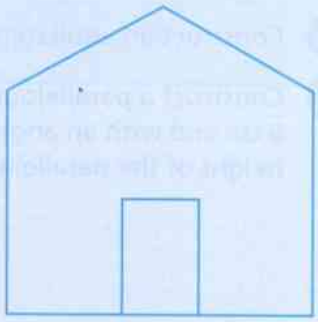
You will generally be given the scale being used, for example, '1 cm represents 20 m'.

Example 2

The diagram shows the front of a kennel. 1 cm on the diagram represents a measurement of 30 cm. Find:

a the actual width of the front

b the actual height of the doorway.



a The actual width of the front is:

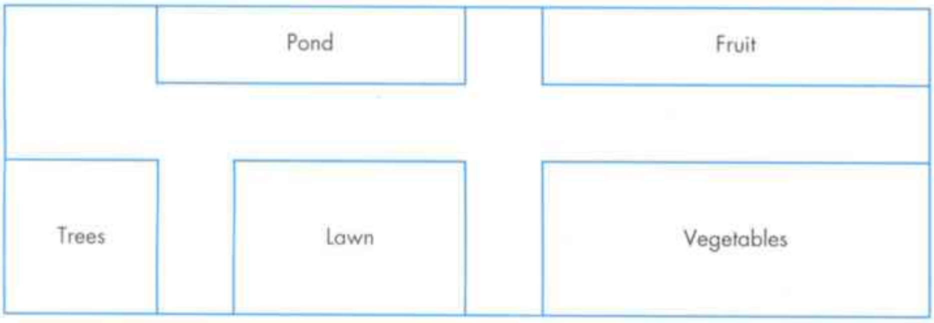
4 cm × 30 = 120 cm

b The actual height of the doorway is:

1.5 cm × 30 = 45 cm

EXERCISE 25B

1 Look at this plan of a garden.

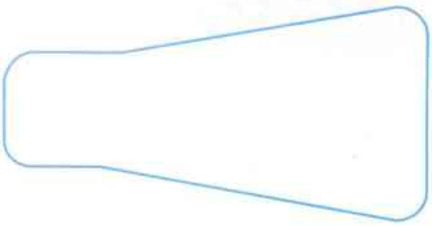


Scale: 1 cm represents 10 m

a State the actual dimensions of each plot of the garden.

b Calculate the actual area of each plot.

2 Below is a plan for a computer mouse mat.



Scale: 1 cm represents 6 cm

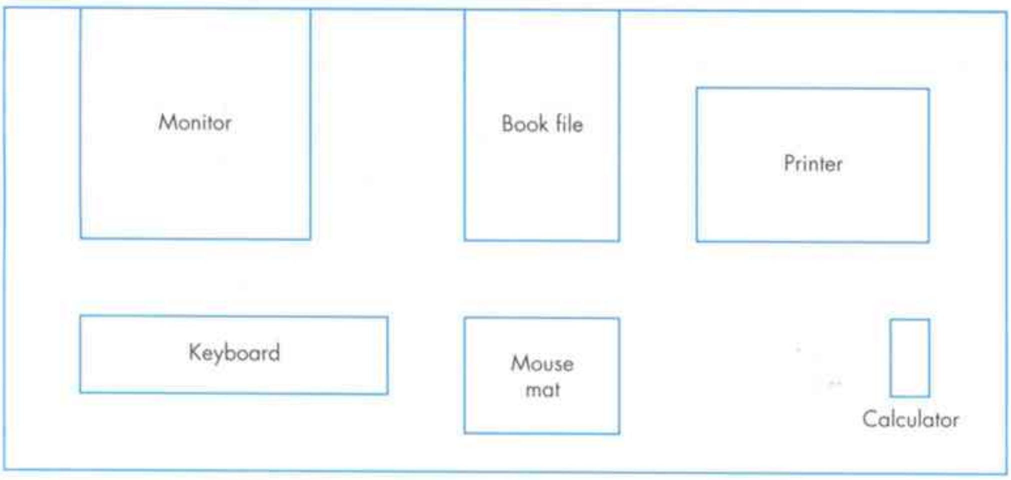
Advice and Tips

Remember to check the scale.

a How long is the actual mouse mat?

b How wide is the narrowest part of the mouse mat?

3 Below is a scale plan of the top of Ahmed's desk, in which 1 cm represents 10 cm.



CORE

What are the actual dimensions of each of these objects?

a monitor

b keyboard

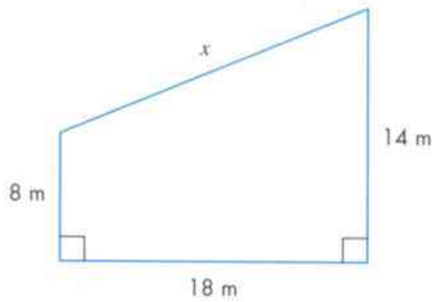
c mouse mat

d book file

e printer

f calculator

4 The diagram shows a sketch of a garden.

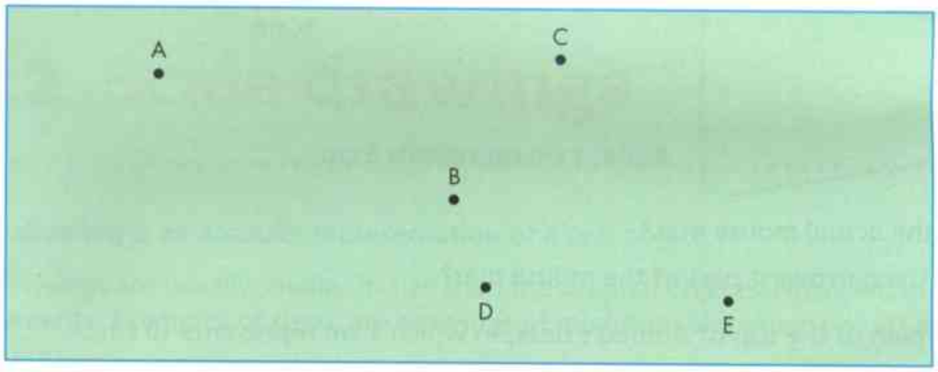


a Make an accurate scale drawing of the garden.

Use a scale of 1 cm to represent 2 m.

b Marie wants to plant flowers along the side marked x on the diagram. The flowers need to be planted 0.5 m apart. Use your scale drawing to work out how many plants she needs.

5 Look at the map below, which is drawn to a scale of 1 cm representing 2 km. Towns are shown with letters.



State these actual distances, correct to the nearest tenth of a kilometre.

a A to B

b B to C

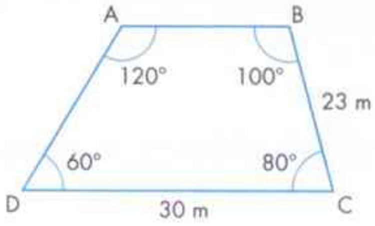
c C to D

d D to E

e E to B

f B to D

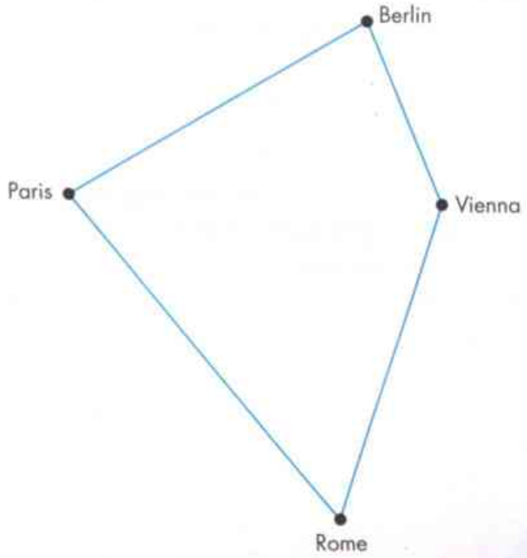
6 This sketch shows the outline of a car park.



a Make a scale drawing in which 1 cm represents 5 m.

b What is the length of AB, in metres?

7 This map is drawn to a scale of 1 cm to 200 km.



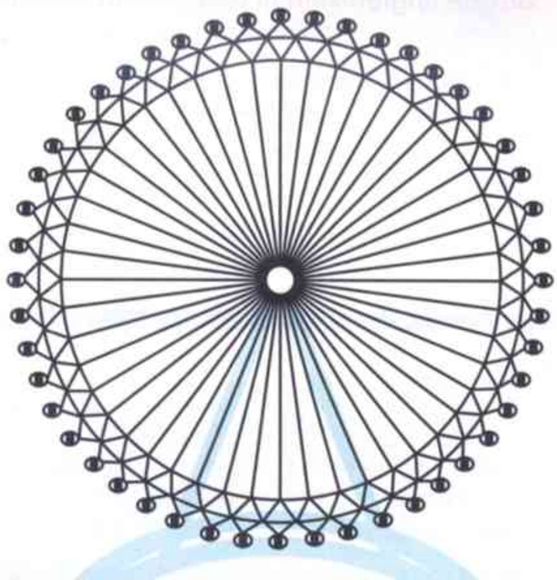
Find the distances between:

a Paris and Berlin

b Paris and Rome

c Rome and Vienna.

8 This is a scale drawing of the Great Beijing Wheel in China.



The height of the wheel is 210 m.

Which of these is the correct scale?

a 1 cm represents 30 cm

b 1 cm represents 7 m

c 1 cm represents 30 m

d 1 cm represents 300 m

Check your progress

Core

- I can measure and draw lines and angles
- I can read and make scale drawings
- I can draw a triangle with a ruler and a pair of compasses when I know the lengths of the sides

Chapter 26

Trigonometry

Topics	Level	Key words
1 Pythagoras' theorem	CORE	hypotenuse, Pythagoras' theorem
2 Trigonometric ratios	CORE	ratio, sine, cosine, tangent, opposite side, adjacent side
3 Calculating angles	CORE	inverse
4 Using sine, cosine and tangent functions	CORE	
5 Which ratio to use	CORE	
6 Applications of trigonometric ratios	EXTENDED	angle of elevation, angle of depression
7 Problems in three dimensions	EXTENDED	
8 Sine and cosine of obtuse angles	EXTENDED	obtuse angle
9 The sine rule and the cosine rule	EXTENDED	sine rule, cosine rule, included angle
10 Using sine to find the area of a triangle	EXTENDED	area sine rule
11 Sine, cosine and tangent of any angle	EXTENDED	

In this chapter you will learn how to:

CORE	EXTENDED
• Apply Pythagoras' theorem and the sine, cosine and tangent ratios for acute angles to the calculation of a side or of an angle of a right-angled triangle. (C6.2 and E6.2)	• Solve trigonometrical problems in two dimensions involving angles of elevation and depression. (E6.2) • Extend sine and cosine values to angles between 90° and 180°. (E6.2) • Recognise, sketch and interpret graphs of simple trigonometric functions. Graph and know the properties of trigonometric functions. Solve simple trigonometric equations for values between 0° and 360°. (E6.3) • Solve problems using the sine and cosine rules for any triangle and the formula area of triangle = $\frac{1}{2}ab\sin C$. (E6.4) • Solve simple trigonometrical problems in three dimensions including angle between a line and a plane. (E6.5)

Why this chapter matters

How can you find the height of a mountain?

How do you draw an accurate map?

How can computers take an image and make it rotate so that you can view it from different directions?

How do Global Positioning Systems (GPS) work?

How can music be produced electronically?

The answer is by using the angles and sides of triangles and the connections between them. This important branch of mathematics is called trigonometry and is used in science, engineering, electronics and everyday life. This chapter gives a brief introduction to trigonometry.

The first major book of trigonometry was a written by an astronomer called Ptolemy, who lived in Alexandria, Egypt, over 1800 years ago.

It has tables of numbers, called 'trigonometric ratios', used in making calculations about the positions of stars and planets.

Trigonometry also helped Ptolemy to make a map of the world he knew. Today you no longer need to look up tables of values of trigonometric ratios because they are programmed into calculators and computers.

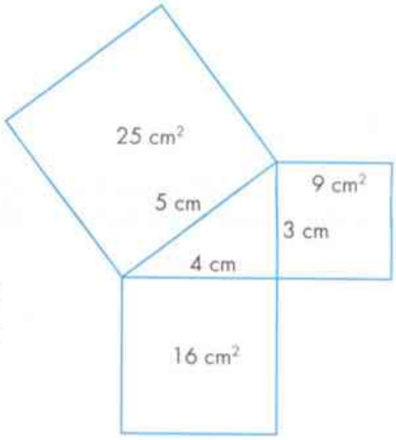
In the 19th century the French mathematician Jean Fourier showed how all musical sounds can be broken down into a combination of tones that can be described by trigonometry. His work makes it possible to imitate the sound of any instrument electronically.

26.1 Pythagoras' theorem

Pythagoras, who was a philosopher as well as a mathematician, was born in 580BCE in Greece. He later moved to Italy, where he established the Pythagorean Brotherhood, which was a secret society devoted to politics, mathematics and astronomy.

This is his famous theorem.

Consider squares being drawn on each side of a right-angled triangle, with sides 3 cm, 4 cm and 5 cm.



The longest side is called the **hypotenuse** and is always opposite the right angle.

Pythagoras' theorem can then be stated as:

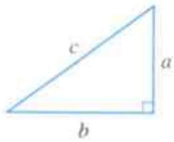
> *For any right-angled triangle, the area of the square drawn on the hypotenuse is equal to the sum of the areas of the squares drawn on the other two sides.*

The usual description is:

> *In any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.*

Pythagoras' theorem is more usually written as a formula:

$c^2 = a^2 + b^2$



Remember that Pythagoras' theorem can only be used in right-angled triangles.

Finding the hypotenuse

Example 1

Find the length of the hypotenuse, marked x on the diagram.

Using Pythagoras' theorem gives:

$x^2 = 8^2 + 5.2^2$ cm^2

$= 64 + 27.04$ cm^2

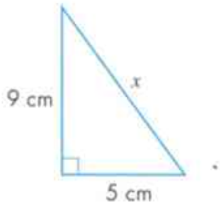
$= 91.04$ cm^2

So $x = \sqrt{91.04} = 9.5$ cm (1 decimal place)

EXERCISE 26A

For each of the triangles in questions **1** to **9**, calculate the length of the hypotenuse, x, giving your answers to 1 decimal place.

1



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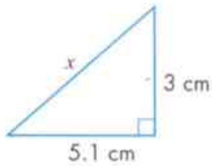

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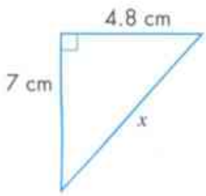

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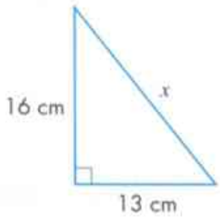

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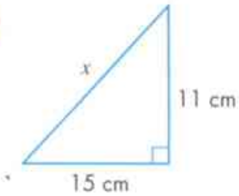

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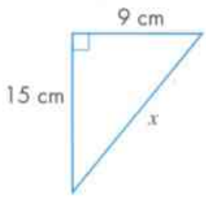

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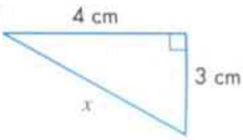

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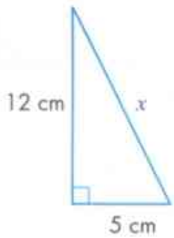
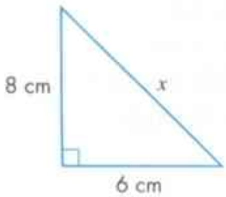


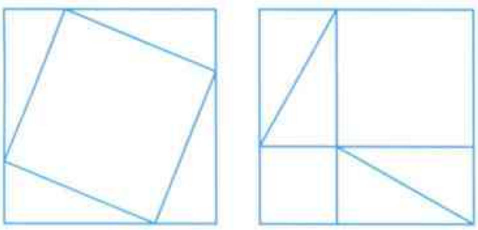
9



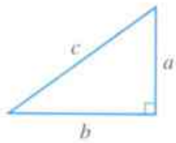
Advice and Tips

In these examples you are finding the hypotenuse. You need to add the squares of the two short sides in every case.

10 How does this diagram show that Pythagoras' theorem is true?



By rearranging the formula for Pythagoras' theorem, you can calculate the length of one of the shorter sides.

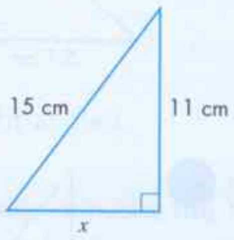


$$c^2 = a^2 + b^2$$

So, $a^2 = c^2 - b^2$ or $b^2 = c^2 - a^2$

Example 2

Find the length of the side labelled x.



x is one of the shorter sides.

So using Pythagoras' theorem gives:

$x^2 = 15^2 - 11^2 \text{ cm}^2$

$= 225 - 121 \text{ cm}^2$

$= 104 \text{ cm}^2$

So $x = \sqrt{104} = 10.2$ cm (1 decimal place)

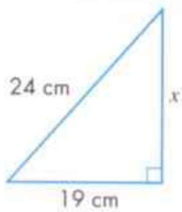
EXERCISE 26B

CORE

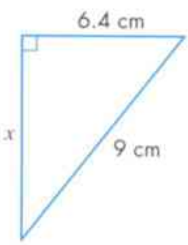
1 For each of these triangles, calculate the length of the side labelled x, giving your answers to 1 decimal place.

a

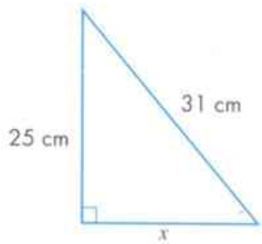
b



c



d

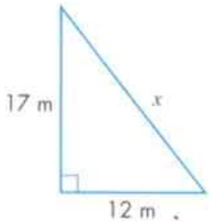


Advice and Tips

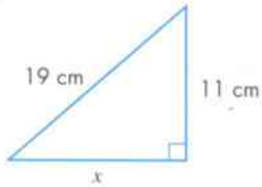
In these examples you are finding a short side. You need to subtract the square of the other short side from the square of the hypotenuse in every case.

2 For each of these triangles, calculate the length labelled x, giving your answers to 1 decimal place.

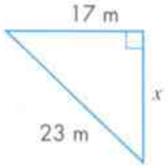
a



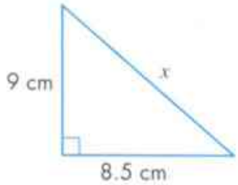
b



c



d

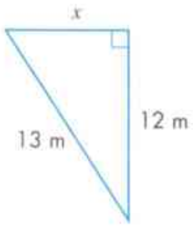


Advice and Tips

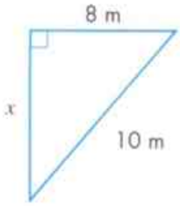
These examples are a mixture. Make sure you combine the squares of the sides correctly.

3 For each of these triangles, calculate the length marked x.

a



b



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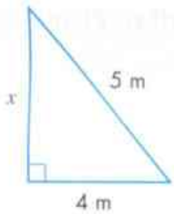
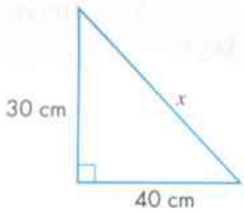


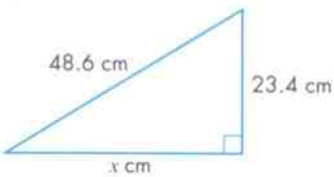
d



4 In question **3** you found sets of three numbers which satisfy $a^2 + b^2 = c^2$.

Can you find any more?

5 Calculate the value of x.

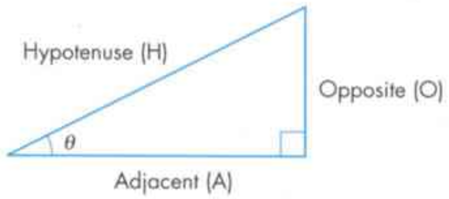


26.2 Trigonometric ratios

In trigonometry you will use three important **ratios** to calculate sides and angles: **sine, cosine** and **tangent**. These ratios are defined in terms of the sides of a right-angled triangle and an angle. The angle is often written as θ.

In a right-angled triangle:

- the side opposite the right angle is called the hypotenuse and is the longest side
- the side opposite the angle θ is called the **opposite side**
- the other side next to both the right angle and the angle θ is called the **adjacent side**.



The sine, cosine and tangent ratios for θ are defined as:

$$\text{sine } \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \text{cosine } \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \text{tangent } \theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

These ratios are usually abbreviated as:

$$\sin \theta = \frac{\text{O}}{\text{H}} \qquad \cos \theta = \frac{\text{A}}{\text{H}} \qquad \tan \theta = \frac{\text{O}}{\text{A}}$$

These abbreviated forms are also used on calculator keys.

Using your calculator

You will need to use a calculator to find trigonometric ratios.

Different calculators work in different ways, so make sure you know how to use your model.

Angles are not always measured in degrees. Sometimes radians or grads are used instead. You do not need to learn about those in your IGCSE course. Calculators can be set to operate in any of these three units, so make sure your calculator is operating in degrees.

Use your calculator to find the sine of 60 degrees.

You will probably press the keys sin 6 0 = in that order, but it might be different on your calculator.

The answer should be 0.8660...

3 cos 57° is a short way of writing 3 × cos 57°.

On most calculators you do not need to use the × button and you can just press the keys in the way it is written: 3 cos 5 7 =

Check to see whether your calculator works this way.

The answer should be 1.63.

Example 3

Find the value of 5.6 sin 30°.

This means 5.6 × sine of 30 degrees.

Remember that you may not need to press the × button.

5.6 sin 30° = 2.8

EXERCISE 26C

CORE

1 Find these values, rounding your answers to 3 significant figures.

a sin 43° **b** sin 56° **c** sin 67.2° **d** sin 90°

2 Find these values, rounding your answers to 3 significant figures.

a cos 43° **b** cos 56° **c** cos 67.2° **d** cos 90°

3 **a** **i** What is sin 35°? **ii** What is cos 55°?

b **i** What is sin 12°? **ii** What is cos 78°?

c **i** What is cos 67°? **ii** What is sin 23°?

d What connects the values in parts **a**, **b** and **c**?

e Copy and complete these sentences.

i sin 15° is the same as cos …

ii cos 82° is the same as sin …

iii sin x is the same as cos …

4 Use your calculator to work out the value of each ratio.

a tan 43° **b** tan 56° **c** tan 67.2° **d** tan 90°

e tan 45° **f** tan 20° **g** tan 22° **h** tan 0°

5 What is so different about tan compared with both sin and cos?

6 Use your calculator to work out the value of each ratio.

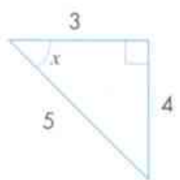
a 4 sin 63° **b** 7 tan 52° **c** 5 tan 80° **d** 9 cos 8°

7 Use your calculator to work out the values of the these ratios.

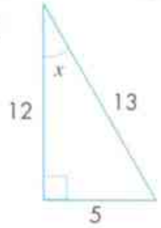
a $\frac{5}{\sin 63°}$ **b** $\frac{6}{\cos 32°}$ **c** $\frac{3}{\tan 64°}$ **d** $\frac{7}{\tan 42°}$

8 Calculate sin x, cos x, and tan x for each triangle. Leave your answers as fractions.

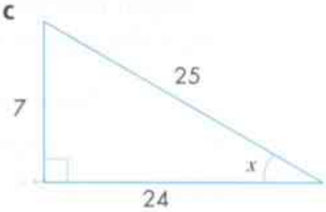
a



b



c



26.3 Calculating angles

What angle has a cosine of 0.6? You can use a calculator to find out.

'The angle with a cosine of 0.6' is written as $\cos^{-1} 0.6$ and is called the '**inverse** cosine of 0.6'.

Find out where $\cos^{-1}$ is on your calculator.

You will probably find it on the same key as cos, but you will need to press SHIFT or INV or 2ndF first.

Look to see if $\cos^{-1}$ is written above the cos key.

Check that $\cos^{-1} 0.6 = 53.1301... = 53.1°$ (1 decimal place)

Check that $\cos 53.1° = 0.600$ (3 decimal places)

Check that you can find the inverse sine and the inverse tangent in the same way.

Example 4

What angle has a sine of $\frac{3}{8}$?

You need to find $\sin^{-1} \frac{3}{8}$.

You could use the fraction button on your calculator or you could calculate $\sin^{-1} (3 \div 8)$.

If you use the fraction key you may not need brackets, or your calculator may put them in automatically.

Try to do it in both of these ways and then use whichever method you prefer.

The answer should be 22.0°.

EXERCISE 26D

CORE

Use your calculator to find the answers to these questions. Give your answers to 1 decimal place.

1 What angles have these sines?

a 0.5 **b** 0.785 **c** 0.64 **d** 0.877 **e** 0.999 **f** 0.707

2 What angles have these cosines?

a 0.5 **b** 0.64 **c** 0.999 **d** 0.707 **e** 0.2 **f** 0.7

3 What angles have these tangents?

a 0.6 **b** 0.38 **c** 0.895 **d** 1.05 **e** 2.67 **f** 4.38

4 What happens when you try to find the angle with a sine of 1.2? What is the largest value of sine you can put into your calculator without getting an error when you ask for the inverse sine? What is the smallest?

5 **a** **i** What angle has a sine of 0.3? (Keep the answer in your calculator memory.)

ii What angle has a cosine of 0.3?

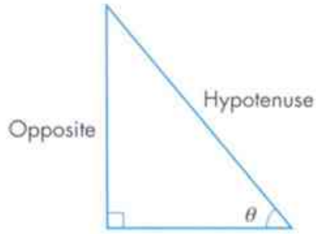
iii Add the two accurate answers of parts **i** and **ii** together.

b Will you always get the same answer to this question, whatever number you start with?

26.4 Using sine, cosine and tangent functions

Sine function

Remember sine $\theta = \dfrac{\text{Opposite}}{\text{Hypotenuse}}$

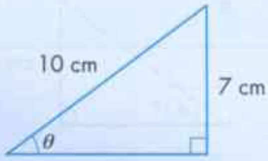


You can use the **sine** ratio to calculate the lengths of sides and angles in right-angled triangles.

Example 5

Find the size of angle θ, given that the opposite side is 7 cm and the hypotenuse is 10 cm.

Draw a diagram. (This is an essential step.)



From the information given, use sine.

$\sin\theta = \dfrac{O}{H} = \dfrac{7}{10} = 0.7$

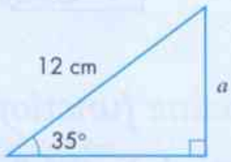
What angle has a sine of 0.7?

To find out, use the inverse sine function on your calculator.

$\sin^{-1} 0.7 = 44.4°$ (1 decimal place)

Example 6

Find the length of the side marked a in this triangle.



Side a is the opposite side, with 12 cm as the hypotenuse, so use sine.

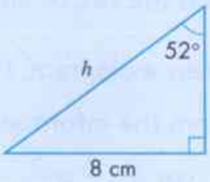
$\sin\theta = \dfrac{O}{H}$

$\sin 35° = \dfrac{a}{12}$

So $a = 12 \sin 35° = 6.88$ cm (3 significant figures)

Example 7

Find the length of the hypotenuse, h, in this triangle.



Note that although the angle is in the other corner, the opposite side is again given. So use sine.

$\sin\theta = \dfrac{O}{H}$

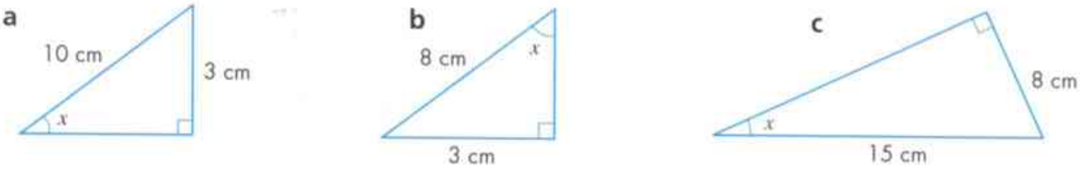
$\sin 52° = \dfrac{8}{h}$

So $h = \dfrac{8}{\sin 52°} = 10.2$ cm (3 significant figures)

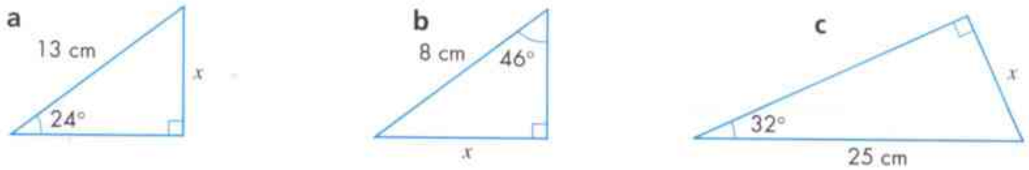
EXERCISE 26E

CORE

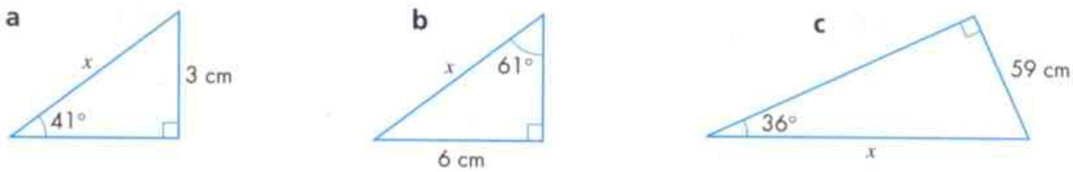
1 Find the size of the angle marked x in each of these triangles.



2 Find the length of the side marked x in each of these triangles.



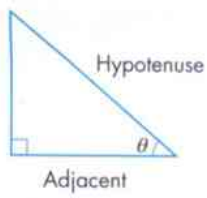
3 Find the length of the side marked x in each of these triangles.



4 Find the length of the side marked x in each of these triangles.

Cosine function

Remember cosine $\theta = \dfrac{\text{Adjacent}}{\text{Hypotenuse}}$

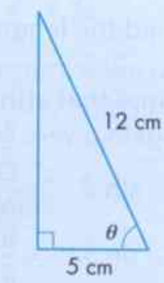


You can use the **cosine** ratio to calculate the lengths of sides and angles in right-angled triangles.

Example 8

Find the size of angle θ, given that the adjacent side is 5 cm and the hypotenuse is 12 cm.

Draw a diagram. (This is an essential step.)



From the information given, use cosine.

$$\cos\,\theta = \frac{A}{H} = \frac{5}{12}$$

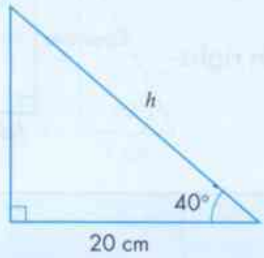
What angle has a cosine of $\frac{5}{12}$?

To find out, use the inverse cosine function on your calculator.

$\cos^{-1} = 65.4°$ (1 decimal place)

Example 9

Find the length of the hypotenuse, h, in this triangle.



The adjacent side is given. So use cosine.

$$\cos \theta = \frac{A}{H}$$

$$\cos 40° = \frac{20}{h}$$

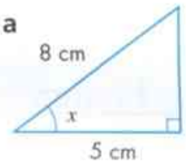
So $h = \dfrac{20}{\cos 40°} = 26.1$ cm (3 significant figures)

EXERCISE 26F

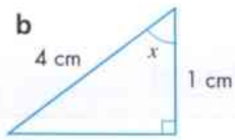
CORE

1 Find the size of the angle marked x in each of these triangles.

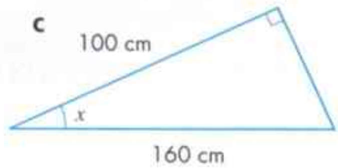
a



b

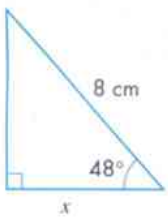


c

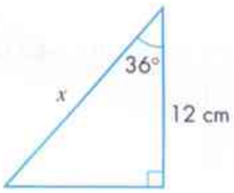


2 Find the length of the side marked x in each of these triangles.

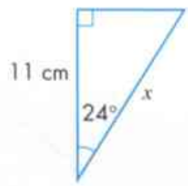
a



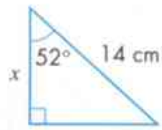
b



c

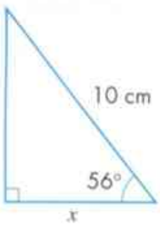


d

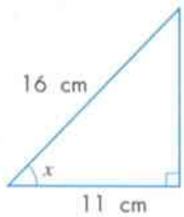


3 Find the value of x in each of these triangles.

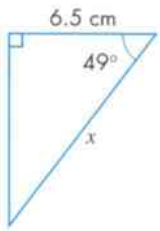
a



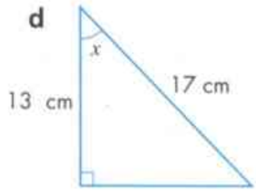
b



c



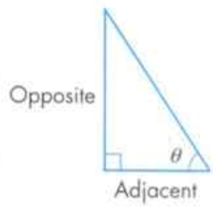
d



Tangent function

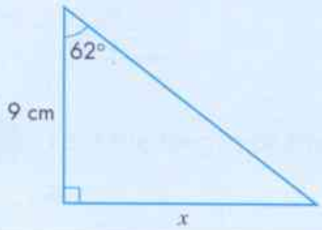
Remember tangent $\theta = \frac{\text{Opposite}}{\text{Adjacent}}$

You can use the **tangent** ratio to calculate the lengths of sides and angles in right-angled triangles.



Example 10

Find the length of the side marked x in this triangle.



The side marked x is the opposite side, with 9 cm as the adjacent side, so use tangent.

$\tan\theta = \frac{\text{O}}{\text{A}}$

$\tan 62° = \frac{x}{9}$

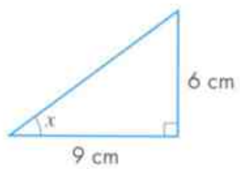
So $x = 9 \tan 62° = 16.9$ cm (3 significant figures)

EXERCISE 26G

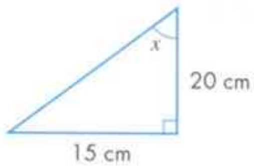
CORE

1 Find the size of the angle marked x in each of these triangles.

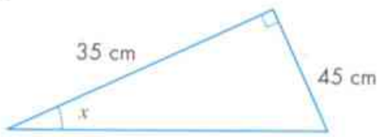
a



b

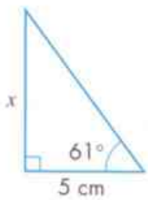


c

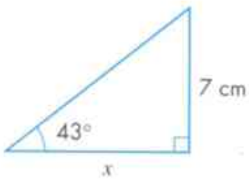


2 Find the length of the side marked x in each of these triangles.

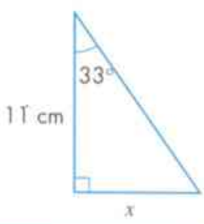
a



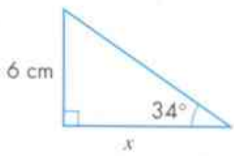
b



c

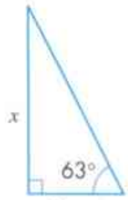


d

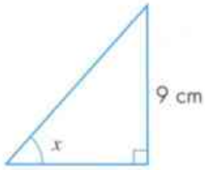


3 Find the value of x in each of these triangles.

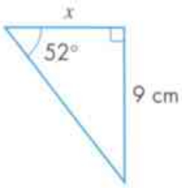
a



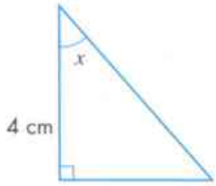
b



c



d



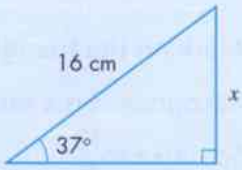
26.5 Which ratio to use

The difficulty with any trigonometric problem is knowing which ratio to use to solve it.

These examples show you how to decide which ratio you need in any given situation.

Example 11

Find the length of the side marked x in this triangle.



Step 1 Identify what information is given and what needs to be found. Namely, x is opposite the angle and 16 cm is the hypotenuse.

Step 2 Decide which ratio to use. Only one ratio uses opposite and hypotenuse: **sine.**

Step 3 Remember $\sin \theta = \frac{O}{H}$

Step 4 Put in the numbers and letters: $\sin 37° = \frac{x}{16}$

Step 5 Rearrange the equation and work out the answer:
$x = 16 \sin 37° = 9.629040371$ cm

Step 6 Give the answer to an appropriate degree of accuracy:
$x = 9.63$ cm (3 significant figures)

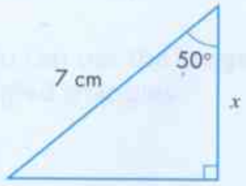
There is no need to write down every step as in Example 11. Step 1 can be done by marking the triangle. Steps 2 and 3 can be done in your head. Steps 4 to 6 are what you write down.

Remember that you must always show evidence of your working. Any reasonable attempt at identifying the sides and using a ratio will probably show that you undertand the method, but only if the fraction is the right way round.

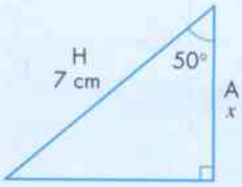
The next examples are set out in a way that requires the *minimum* amount of working but gets *maximum* results.

Example 12

Find the length of the side marked x in this triangle.



Mark on the triangle the side you know (H) and the side you want to find (A).



Recognise it is a **cosine** problem because you have A and H.

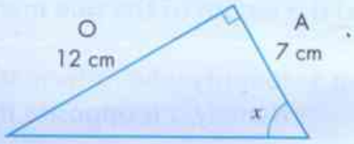
So $\cos 50° = \frac{x}{7}$

$x = 7 \cos 50° = 4.50$ cm (3 significant figures)

Example 13

Find the size of the angle marked x in this triangle.

Mark on the triangle the sides you know.



Recognise it is a **tangent** problem because you have O and A.

So $\tan x = \frac{12}{7}$

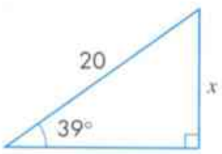
$x = \tan^{-1} \frac{12}{7} = 59.7°$ (1 decimal place)

EXERCISE 26H

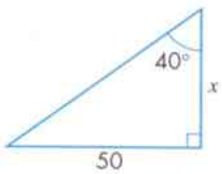
CORE

1 Find the length marked x in each of these triangles.

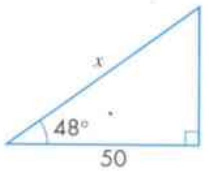
a



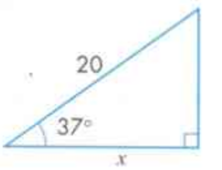
b



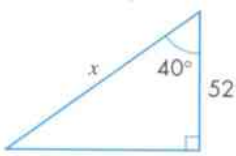
c



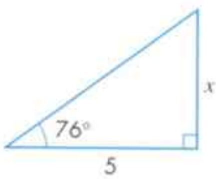
d



e

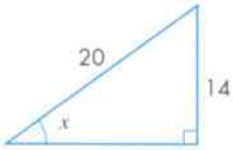


f

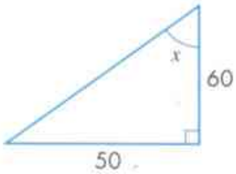


2 Find the size of the angle marked x in each of these triangles.

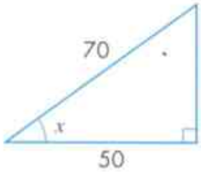
a



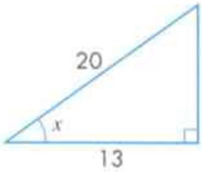
b



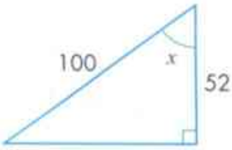
c



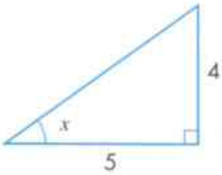
d



e

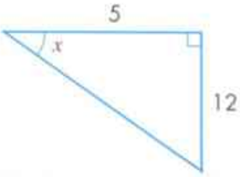


f

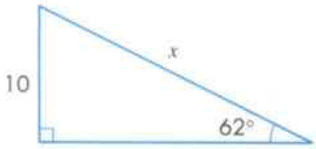


3 Find the value of the angle or length marked x in each of these triangles.

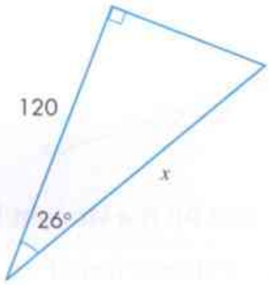
a



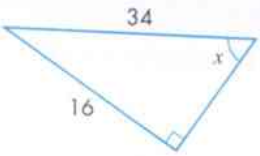
b



c

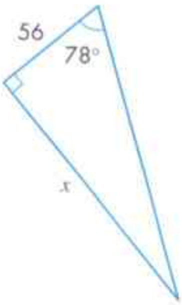


d

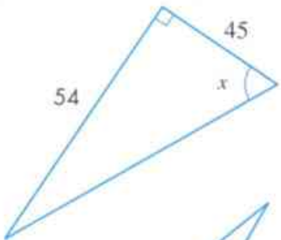


e

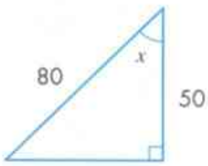
f



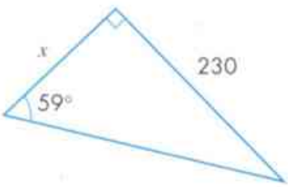
g



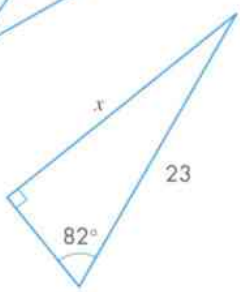
h



i

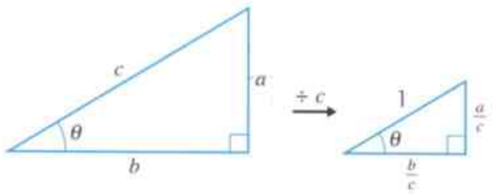


j



CORE

4 a How does this diagram show that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$?

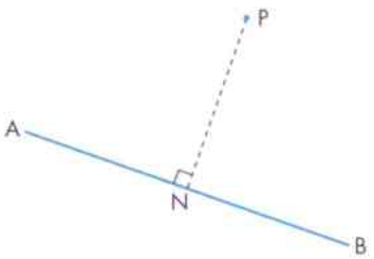


b How does the diagram show that $(\sin\theta)^2 + (\cos\theta)^2 = 1$?

c Choose a value for θ and check the two results in parts **a** and **b** are true.

26.6 Applications of trigonometric ratios

E



P is a point and AB is a straight line.

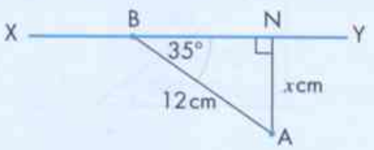
The shortest distance from P to AB is the length of PN where PN is perpendicular to AB.

The angle PNA is a right angle.

Example 14

Find the shortest distance from the point A to the line XY.

Draw a right-angled triangle.

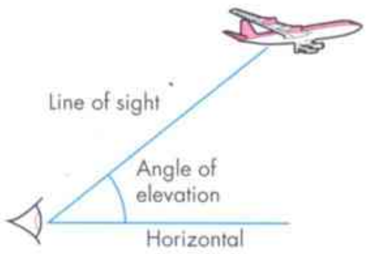


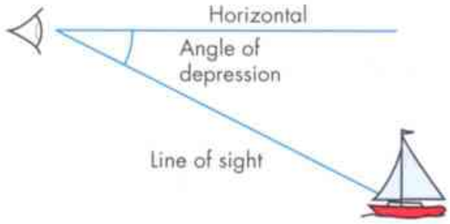
You want to find x.

$\sin 35° = \dfrac{x}{12}$ and so $x = 12 \sin 35° = 6.88$ cm (to 3 s.f.)

When you look *up* at an aircraft in the sky, the angle through which your line of sight turns, from looking straight ahead (the horizontal), is called the **angle of elevation**.

When you are standing on a high point and look *down* at a boat, the angle through which your line of sight turns, from looking straight ahead (the horizontal), is called the **angle of depression**.





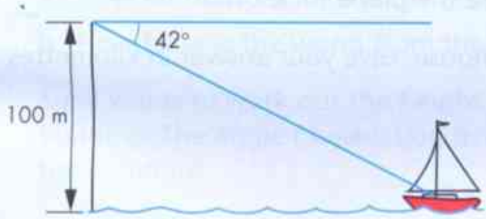
Example 15

From the top of a vertical cliff, 100 m high, Ali sees a boat out at sea. The angle of depression from Ali to the boat is 42°. How far from the base of the cliff is the boat?

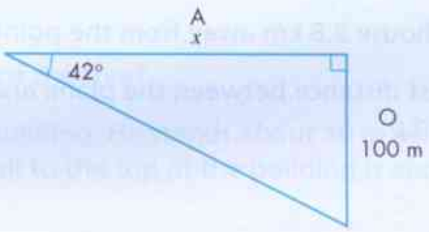
The diagram of the situation is shown in figure **i**.

From this, you get the triangle shown in figure **ii**.

i



ii



From figure **ii**, you see that this is a tangent problem.

So $\tan 42° = \dfrac{100}{x}$

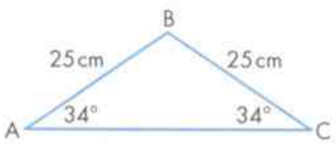
$$x = \frac{100}{\tan 42°}$$

$= 111$ m (3 significant figures)

EXERCISE 26I

In these questions, give any answers involving angles to the nearest degree.

1



ABC is an isosceles triangle.

Find the shortest distance from **B** to **AC**.

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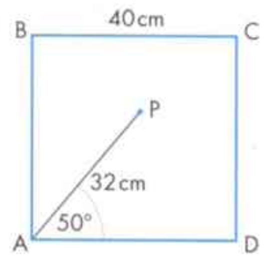
2



ABCD is a square. Each side is 40 cm.

P is a point inside the square.

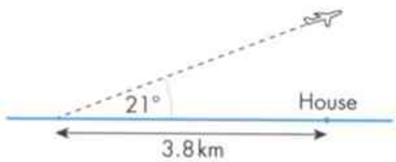
Calculate the distance from P to

a AD

b AB

c CD

3

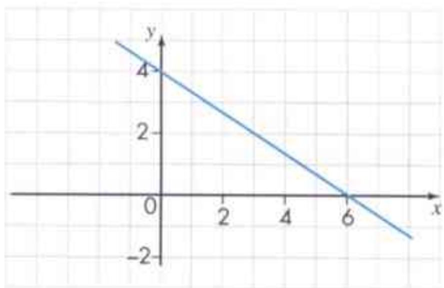


A plane takes off and rises at an angle of 21° to the ground.

It flies over a house 3.8 km away from the point where the plane took off.

Find the closest distance between the plane and the house. Give your answer in kilometres rounded to 2 d.p.

4



This is a graph of the line with equation $2x + 3y = 12$

Find the shortest distance from the origin to the line.

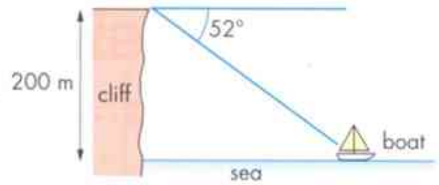
5 Erik sees an aircraft in the sky. The aircraft is at a horizontal distance of 25 km from Erik. The angle of elevation is 22°.

How high is the aircraft?

6 An aircraft is flying at an altitude of 4000 m and is 10 km from the airport. If a passenger can see the airport, what is the angle of depression?

7 A man standing 200 m from the base of a television transmitter looks at the top of it and notices that the angle of elevation of the top is 65°.

How high is the tower?

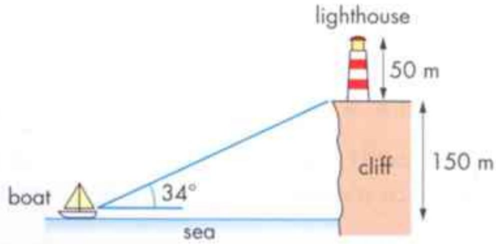
8 

a From the top of a vertical cliff, 200 m high, a boat has an angle of depression of 52°. How far from the base of the cliff is the boat?

b The boat now sails away from the cliff so that the distance is doubled. Does that mean that the angle of depression is halved?

Give a reason for your answer.

9 From a boat, the angle of elevation of the foot of a lighthouse on the edge of a cliff is 34°.



a If the cliff is 150 m high, how far from the base of the cliff is the boat?

b If the lighthouse is 50 m high, what would be the angle of elevation of the top of the lighthouse from the boat?

10 A bird flies from the top of a 12 m tall tree, at an angle of depression of 34°, to catch a worm on the ground.

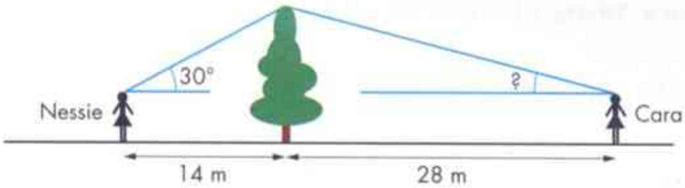
a How far does the bird actually fly?

b How far was the worm from the base of the tree?

11 Sunil wants to work out the height of a building. He stands about 50 m away from the building. The angle of elevation from Sunil to the top of the building is about 15°. How tall is the building?

12 The top of a ski run is 100 m above the finishing line. The run is 300 m long. What is the angle of depression of the ski run?

13 Nessie and Cara are standing on opposite sides of a tree.



Nessie is 14 m away and the angle of elevation of the top of the tree is 30°.

Cara is 28 m away. She says the angle of elevation for her must be 15° because she is twice as far away.

Is she correct?

What do you think the angle of elevation is?

Chapter 26 . Topic 7

26.7 Problems in three dimensions

E

To find the value of an angle or side in a three-dimensional figure you need to find a right-angled triangle in the figure. This triangle also has to include two known values that you can use in the calculation.

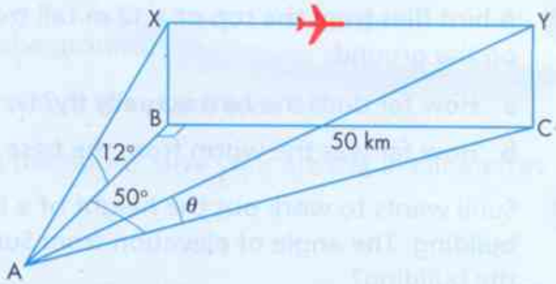
You must redraw this triangle separately as a plain, right-angled triangle. Add the known values and the unknown value you want to find. Then use the trigonometric ratios and Pythagoras' theorem to solve the problem.

Example 16

A, B and C are three points at ground level. They are in the same horizontal plane. C is 50 km east of B. B is north of A. C is on a bearing of 050° from A.

An aircraft, flying east, passes over B and over C at the same height. When it passes over B, the angle of elevation from A is 12°. Find the angle of elevation of the aircraft from A when it is over C.

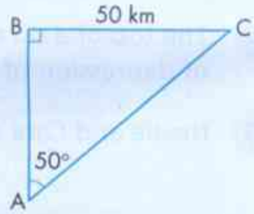
First, draw a diagram containing all the known information.



Next, use the right-angled triangle ABC to calculate AB and AC.

$AB = \frac{50}{\tan 50°} = 41.95$ km (4 significant figures)

$AC = \frac{50}{\sin 50°} = 65.27$ km (4 significant figures)



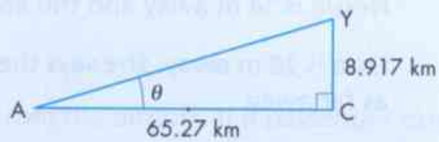
Then use the right-angled triangle ABX to calculate BX, and hence CY.

$BX = 41.95 \tan 12° = 8.917$ km (4 significant figures)

Finally, use the right-angled triangle ACY to calculate the required angle of elevation, θ.

$\tan \theta = \frac{8.917}{65.27} = 0.1366$

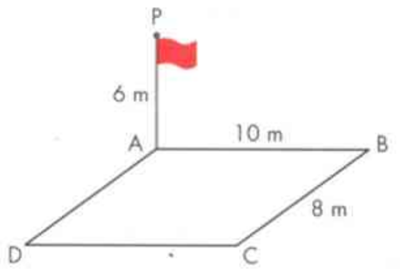
$\Rightarrow \theta = \tan^{-1} 0.1366 = 7.8°$ (1 decimal place)



Always write down intermediate working values to at least 4 significant figures, or use the answer on your calculator display to avoid inaccuracies in the final answer.

EXERCISE 26J

1

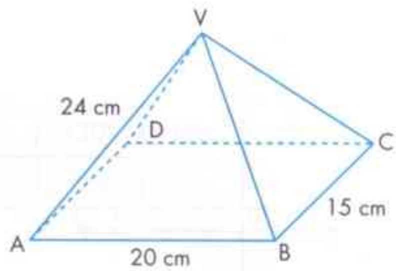


A vertical flagpole AP stands at the corner of a rectangular courtyard ABCD.

Calculate the angle of elevation of P from C.

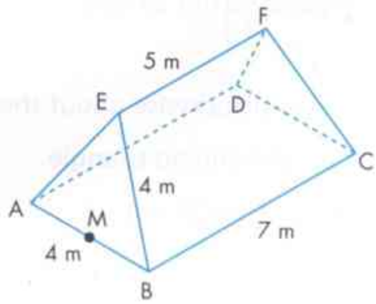
2 The diagram shows a pyramid. The base is a horizontal rectangle ABCD, 20 cm by 15 cm. The length of each sloping edge is 24 cm. The apex, V, is over the centre of the rectangular base. Calculate:

- **a** the length of the diagonal AC
- **b** the size of the angle VAC
- **c** the height of the pyramid.



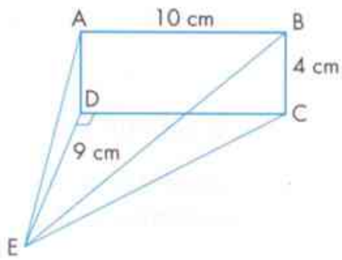
3 The diagram shows the roof of a building. The base ABCD is a horizontal rectangle 7 m by 4 m. The triangular ends are equilateral triangles. Each side of the roof is an isosceles trapezium. The length of the top of the roof, EF, is 5 m. Calculate:

- **a** the length EM, where M is the midpoint of AB
- **b** the size of angle EBC
- **c** the size of the angle between EM and the base ABCD.



4 ABCD is a vertical rectangular plane. EDC is a horizontal triangular plane. Angle CDE = 90°, AB = 10 cm, BC = 4 cm and ED = 9 cm. Calculate:

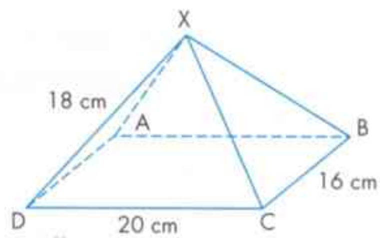
- **a** angle AED
- **b** angle DEC
- **c** EC
- **d** angle BEC.



5 In the diagram, XABCD is a pyramid with a rectangular base.

Revina says that the angle between the edge XD and the base ABCD is 56.3°.

Work out the correct answer to show that Revina is wrong.



Chapter 26 . Topic 8

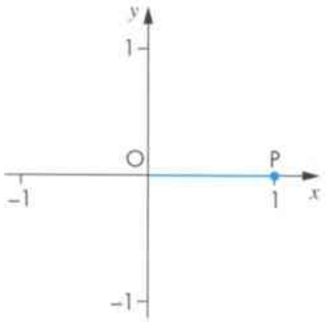
26.8 Sine and cosine of obtuse angles

E

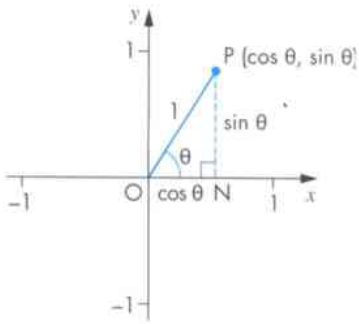
So far you have only used sines and cosines in right-angled triangles.
A calculator also gives the sine and cosine of **obtuse angles**.
Check that sin 115° = 0.906 and cos 115° = −0.423.

You cannot have a right-angled triangle with an obtuse angle so how can you calculate sin and cos of an obtuse angle?

Imagine a rod OP lying on the x-axis as shown.



It rotates anticlockwise about the origin O.



ONP is a right-angled triangle.

The hypotenuse OP = 1

If the angle that OP makes with the x-axis is θ:

The adjacent side ON = OP × cos θ = 1 × cos θ = cos θ

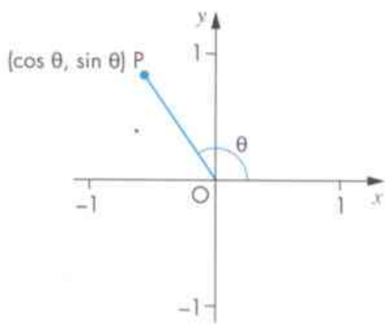
So the x-coordinate of P is cos θ

Similarly:

The opposite side NP = OP × sin θ = 1 × sin θ = sin θ

So the y-coordinate of P is sin θ
Therefore the coordinates of P are (cos θ, sin θ).

Imagine that OP continues to rotate so that angle θ becomes obtuse.



You can still define cos θ and sin θ as the coordinates of P.

You can see from the diagram that the x-coordinate for P must now be negative.

The y-coordinate is still positive.

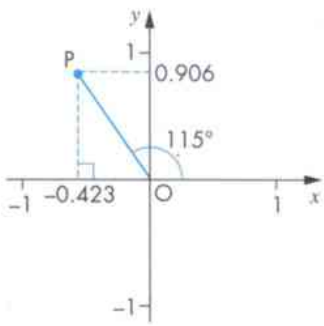
For example, if θ = 115°, the diagram looks like this:

You can see that the angle adjoining θ on the x-axis must have the same sine as θ. When adjacent angles add up to 180° their sines are the same. So:

If angle θ is obtuse then $\sin\theta = \sin(180° - \theta)$

Their cosines have the same numeric value but the cosine of the obtuse angle is negative:

If angle θ is obtuse $\cos\theta = -\cos(180° - \theta)$



For example, if $\theta = 115°$ then $180° - \theta = 65°$

$\sin 115° = \sin 65° = 0.906$

$\cos 115° = -\cos 65° = -0.423$

EXERCISE 26K

1 a Copy and complete this table.

Angle	10°	30°	50°	85°	90°	95°	130°	150°	170°
Sine		0.5			1				

b Draw a graph of $y = \sin x$ for $0 \leqslant x \leqslant 180°$.

c Describe the symmetry of the graph.

d Find two examples from the table to show that if two angles add up to 180 degrees they have the same sine.

2 Find two angles that have a sine of 0.5.

3 Find two angles that have a sine of 0.72.

4 x is an obtuse angle and $\sin x = 0.84$. Find the value of x.

5 a Copy and complete this table.

Angle	15°	35°	60°	80°	90°	100°	120°	145°	165°
Cosine									

b Draw a graph of $y = \cos x$ for $0 \leqslant x \leqslant 180°$.

c Describe the symmetry of the graph.

6 Find the size of each of these angles.

a $\cos^{-1}0.85$ b $\cos^{-1}(-0.85)$

c $\cos^{-1}(-0.5)$ d $\cos^{-1}0$

e $\cos^{-1}0.125$ f $\cos^{-1}(-0.125)$

7 Solve these equations where $0 \leqslant x \leqslant 180°$. Give your answers to the nearest degree. There may be more than one solution.

a $\cos x = 0.6$ b $\cos x = -0.25$

c $\sin x = \frac{3}{4}$ d $\sin x = 1$

e $\cos x = 0$ f $\sin x = 0.95$

g $\sin x = 2$ h $\sin x = \cos x$

EXTENDED

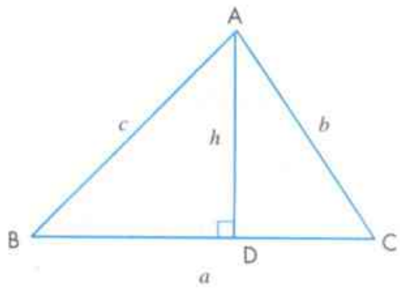
26.9 The sine rule and the cosine rule

E

Any triangle has six measurements: three sides and three angles. To find any unknown angles or sides), you need to know at least three of the measurements. Any combination of three measurements – except that of all three angles – is enough to work out the rest.

When you need to find the value of a side or an angle in a triangle that contains no right angle, you can use one of two rules, depending on what you know about the triangle. These are the **sine rule** and the **cosine rule**.

The sine rule



Take a triangle ABC and draw the perpendicular from A to the opposite side BC.

From right-angled triangle ADB, $h = c \sin B$

From right-angled triangle ADC, $h = b \sin C$

Therefore,

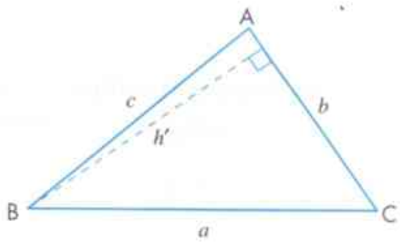
$$c \sin B = b \sin C$$

which can be rearranged to give:

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

By drawing a perpendicular from each of the other two vertices to the opposite side (or by algebraic symmetry), you see that:

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \text{and that} \quad \frac{a}{\sin A} = \frac{b}{\sin B}$$

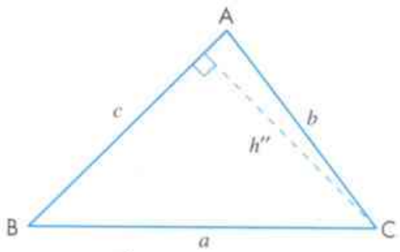


These are usually combined in the form:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

which can be inverted to give:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



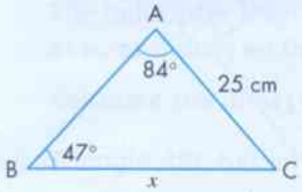
Remember, when using the sine rule: take each side in turn, divide it by the sine of the angle opposite and then equate the results.

Note:

- When you are calculating a *side*, use the rule with the *sides on top*.
- When you are calculating an *angle*, use the rule with the *sines on top*.

Example 17

In triangle ABC, find the value of x.



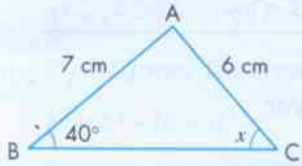
Use the sine rule with sides on top, which gives:

$$\frac{x}{\sin 84°} = \frac{25}{\sin 47°}$$

$$\Rightarrow x = \frac{25 \sin 84°}{\sin 47°} = 34.0 \text{ cm} \text{ (3 significant figures)}$$

Example 18

In triangle ABC, find the value of the acute angle x.



Use the sine rule with sines on top, which gives:

$$\frac{\sin x}{7} = \frac{\sin 40°}{6}$$

$$\Rightarrow \sin x = \frac{7 \sin 40°}{6} = 0.7499$$

$$\Rightarrow x = \sin^{-1} 0.7499 = 48.6° \text{ (3 significant figures)}$$

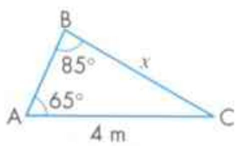
The sine rule works even if the triangle has an obtuse angle, because you can find the sine of an obtuse angle.

EXERCISE 26L

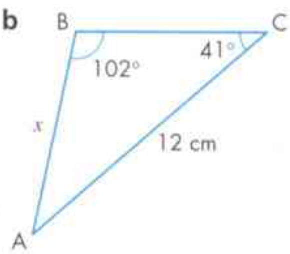
EXTENDED

1 Find the length of the side labelled x in each of these triangles.

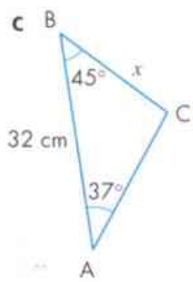
a



b



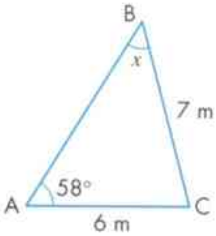
c



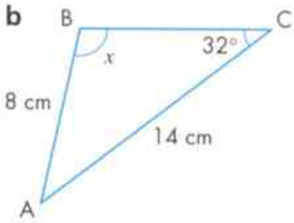
EXTENDED

2 Find the size of the angle labelled x in each of these triangles.

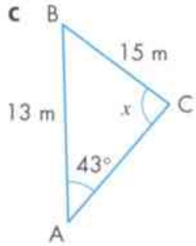
a



b



c



3 To find the height of a tower standing on a small hill, Maria made some measurements (see diagram).

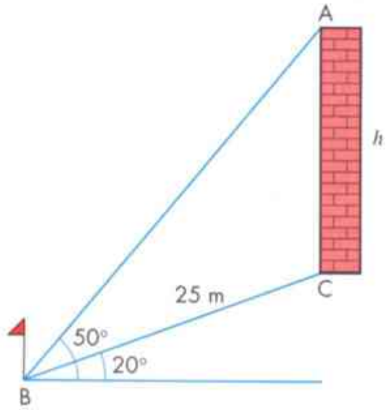
From a point B, the angle of elevation of C is 20°, the angle of elevation of A is 50° and the distance BC is 25 m.

a Calculate these angles.

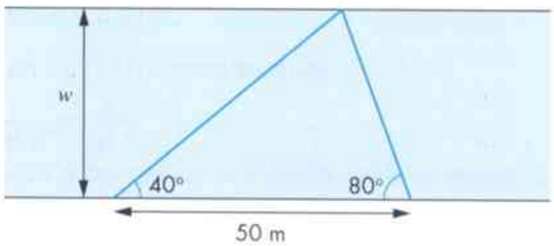
i ABC

ii BAC

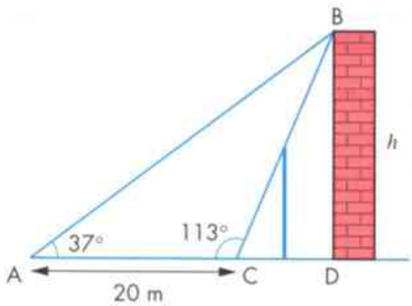
b Using the sine rule and triangle ABC, calculate the height h of the tower.



4 Use the information on this sketch to calculate the width, w, of the river.



5



An old building is unsafe and is protected by a fence. A company is going to demolish the building and has to work out its height BD, marked h on the diagram.

Calculate the value of h, using the given information.

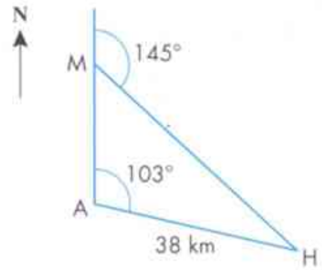
6 A mass is hung from a horizontal beam using two strings. The shorter string is 2.5 m long and makes an angle of 71° with the horizontal. The longer string makes an angle of 43° with the horizontal. What is the length of the longer string?

7 A rescue helicopter is based at an airfield at A.

It is sent out to rescue a man from a mountain at M, due north of A.

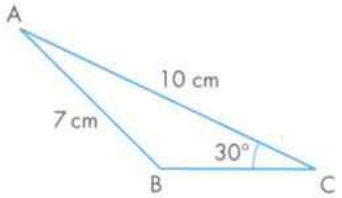
The helicopter then flies on a bearing of 145° to a hospital at H, as shown on the diagram.

Calculate the direct distance from the mountain to the hospital.



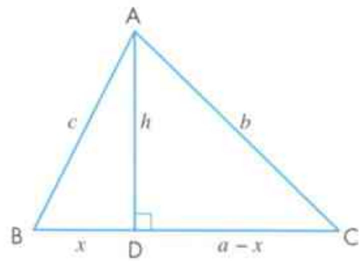
8 Triangle ABC has an obtuse angle at B.

Calculate the size of angle ABC.



The cosine rule

Take the triangle, shown on the right, where D is the foot of the perpendicular to BC from A. The length of BD is x.



Using Pythagoras' theorem on triangle BDA:

$$h^2 = c^2 - x^2$$

Using Pythagoras' theorem on triangle ADC:

$$h^2 = b^2 - (a - x)^2$$

Therefore,

$$c^2 - x^2 = b^2 - (a - x)^2$$

$$c^2 - x^2 = b^2 - a^2 + 2ax - x^2$$

$$\Rightarrow c^2 = b^2 - a^2 + 2ax$$

From triangle BDA, $x = c \cos B$.

So:

$$c^2 = b^2 - a^2 + 2ac \cos B$$

Rearranging gives:

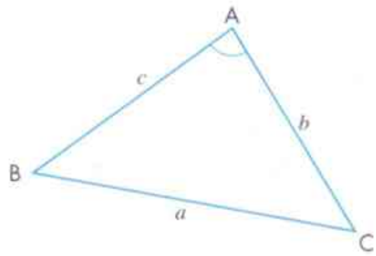
$$b^2 = a^2 + c^2 - 2ac \cos B$$

By algebraic symmetry:

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \text{and} \quad c^2 = a^2 + b^2 - 2ab \cos C$$

This is the cosine rule, which can be best remembered from the diagram on the right, where:

$$a^2 = b^2 + c^2 - 2bc \cos A$$



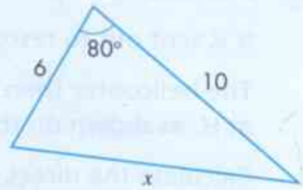
Note the symmetry of the rule and how the rule works using two adjacent sides and the angle between them (the **included angle**).

The formula can be rearranged to find any of the three angles.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 19

Find the value of x in this triangle.



By the cosine rule:

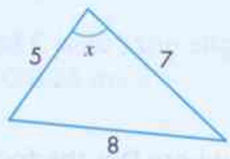
$$x^2 = 6^2 + 10^2 - 2 \times 6 \times 10 \times \cos 80°$$

$$x^2 = 115.16$$

$$\Rightarrow x = 10.7 \text{ (3 significant figures)}$$

Example 20

Find the value of x in this triangle.



By the cosine rule:

$$\cos x = \frac{5^2 + 7^2 - 8^2}{2 \times 5 \times 7} = 0.1428$$

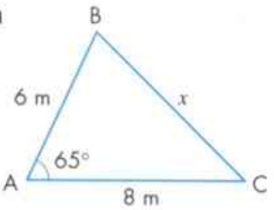
$$\Rightarrow x = 81.8° \text{ (3 significant figures)}$$

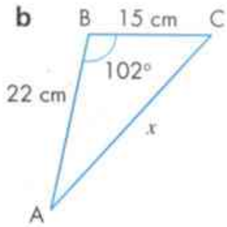
It is possible to find the cosine of an angle that is greater than 90°. For example, cos 120° = −0.5.

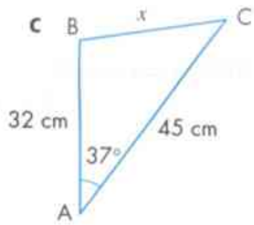
EXERCISE 26M

EXTENDED

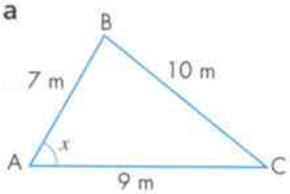
1 Find the length of the side marked x in each of these triangles.

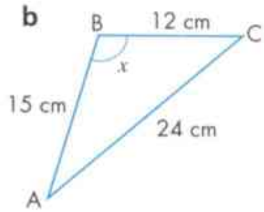
a


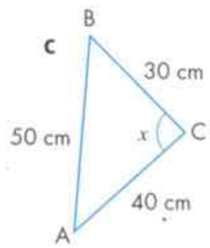
b


c


2 Find the angle x in each of these triangles.

a


b


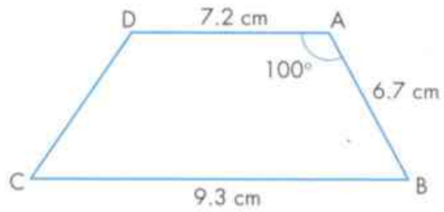
c


d Explain the significance of the answer to part **c**.

3 In triangle ABC, AB = 5 cm, BC = 6 cm and angle ABC = 55°. Find AC.

4 A triangle has two sides of length 40 cm and an angle of 110°. Work out the length of the third side of the triangle.

5 The diagram shows a trapezium ABCD. AB = 6.7 cm, AD = 7.2 cm, CB = 9.3 cm and angle DAB = 100°.



Calculate:

a the length DB **b** angle DBA

c angle DBC **d** the length DC

6 A ship sails from a port on a bearing of 050° for 50 km then turns on a bearing of 150° for 40 km. A crewman is taken ill, so the ship drops anchor. What course and distance should a rescue helicopter from the port fly to reach the ship in the shortest possible time?

7 The three sides of a triangle are given as $3a$, $5a$ and $7a$. Calculate the smallest angle in the triangle.

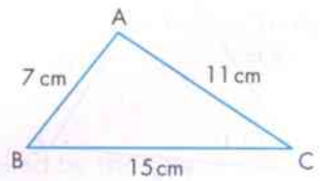
8 Two ships, X and Y, leave a port at 9 am.

Ship X travels at an average speed of 20 km/h on a bearing of 075° from the port.

Ship Y travels at an average speed of 25 km/h on a bearing of 130° from the port.

Calculate the distance between the two ships at 11 am.

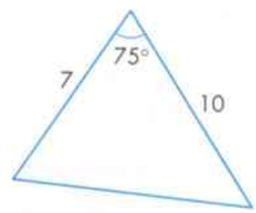
9 Calculate the size of the largest angle in the triangle ABC.



Choosing the correct rule

When finding the unknown sides and angles in a triangle, there are several situations that can occur.

Two sides and the included angle

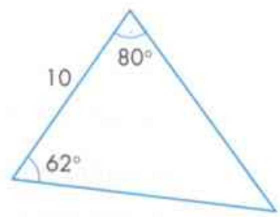


1 Use the cosine rule to find the third side.

2 Use the sine rule to find either of the other angles.

3 Use the sum of the angles in a triangle to find the third angle.

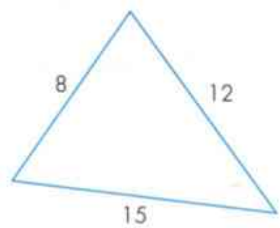
Two angles and a side



1 Use the sum of the angles in a triangle to find the third angle.

2, 3 Use the sine rule to find the other two sides.

Three sides



1 Use the cosine rule to find one angle.

2 Use the sine rule to find another angle.

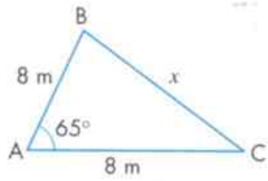
3 Use the sum of the angles in a triangle to find the third angle.

EXERCISE 26N

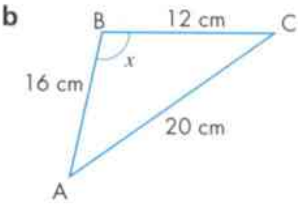
EXTENDED

1 Find the length or angle x in each of these triangles.

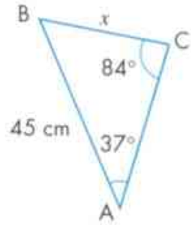
a



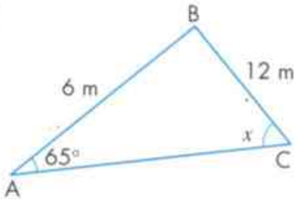
b



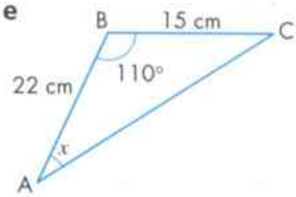
c



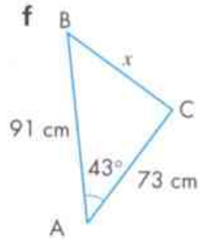
d



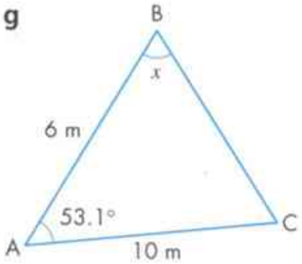
e



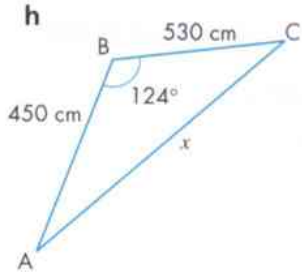
f



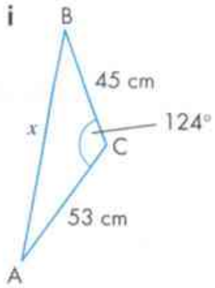
g



h



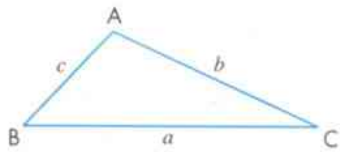
i



2 The hands of a clock have lengths 3 cm and 5 cm. Find the distance between the tips of the hands at 4 o'clock.

3 A helicopter is seen hovering at a point which is in the same vertical plane as two towns, X and F, which are on the same level. Its distances from X and F are 8.5 km and 12 km respectively. The angle of elevation of the helicopter when observed from F is 43°. Calculate the distance between the two towns.

4 Triangle ABC has sides with lengths a, b and c, as shown in the diagram.



a What can you say about the angle BAC, if $b^2 + c^2 - a^2 = 0$?

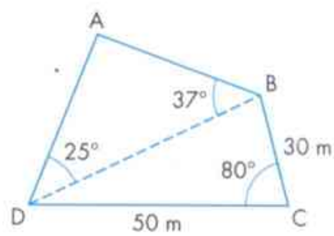
b What can you say about the angle BAC, if $b^2 + c^2 - a^2 \geqslant 0$?

c What can you say about the angle BAC, if $b^2 + c^2 - a^2 \leqslant 0$?

5 The diagram shows a sketch of a field ABCD.

A farmer wants to put a new fence round the perimeter of the field.

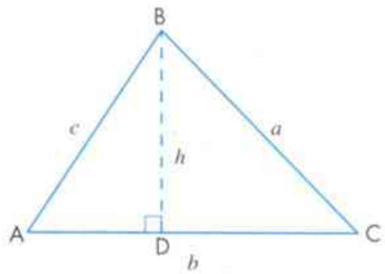
Calculate the perimeter of the field.



26.10 Using sine to find the area of a triangle

E

In triangle ABC, the vertical height is BD and the base is AC.



Let $BD = h$ and $AC = b$, then the **area** of the triangle is given by:

$$\tfrac{1}{2} \times AC \times BD = \tfrac{1}{2}bh$$

However, in triangle BCD:

$$h = BC \sin C = a \sin C$$

where $BC = a$.

Substituting into $\tfrac{1}{2}bh$ gives:

$$\tfrac{1}{2}b \times (a \sin C) = \tfrac{1}{2}ab \sin C$$

as the area of the triangle.

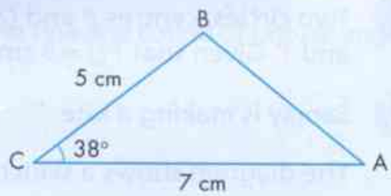
By taking the perpendicular from A to its opposite side BC, and the perpendicular from C to its opposite side AB, you can show that the area of the triangle is also given by:

$$\tfrac{1}{2}ac \sin B \quad \text{and} \quad \tfrac{1}{2}bc \sin A$$

Note the pattern: the area is given by the product of two sides multiplied by the sine of the included angle. This is the **area sine rule**. Starting from any of the three forms, you can use the sine rule to establish the other two.

Example 21

Find the area of triangle ABC.



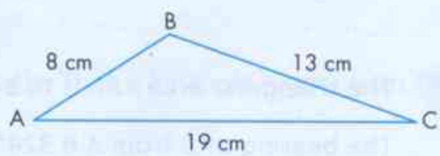
Area $= \tfrac{1}{2}ab \sin C$

Area $= \tfrac{1}{2} \times 5 \times 7 \times \sin 38°$

$= 10.8\ \text{cm}^2$ (3 significant figures)

Example 22

Find the area of triangle ABC.



You have all three sides but no angle. So first you must find an angle in order to apply the area sine rule.

Use the cosine rule to find angle C.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{13^2 + 19^2 - 8^2}{2 \times 13 \times 19} = 0.9433 \quad \Rightarrow C = \cos^{-1} 0.9433 = 19.4°$$

(Keep the exact value in your calculator memory.)

Now apply the area sine rule.

$\frac{1}{2}ab \sin C = \frac{1}{2} \times 13 \times 19 \times \sin 19.4°$

$= 41.0 \text{ cm}^2$ (3 significant figures)

EXERCISE 26O

EXTENDED

1 Find the area of each of these triangles.

- **a** Triangle ABC with $BC = 7$ cm, $AC = 8$ cm and angle $ACB = 59°$
- **b** Triangle ABC with angle $BAC = 86°$, $AC = 6.7$ cm and $AB = 8$ cm
- **c** Triangle PQR with $QR = 27$ cm, $PR = 19$ cm and angle $QRP = 109°$
- **d** Triangle XYZ with $XY = 231$ cm, $XZ = 191$ cm and angle $YXZ = 73°$
- **e** Triangle LMN with $LN = 63$ cm, $LM = 39$ cm and angle $NLM = 85°$

2 The area of triangle ABC is 27 cm^2. If $BC = 14$ cm and angle $BCA = 115°$, find the length of AC.

3 The area of triangle LMN is 113 cm^2, $LM = 16$ cm and $MN = 21$ cm. Angle LMN is acute. Calculate these angles.

- **a** Angle LMN
- **b** Angle MNL

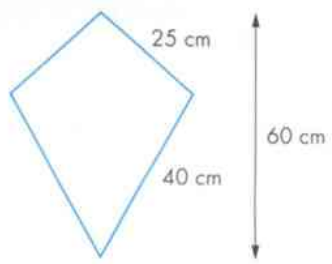
4 A board is in the shape of a triangle with sides 60 cm, 70 cm and 80 cm. Find the area of the board.

5 Two circles, centres P and Q, have radii of 6 cm and 7 cm respectively. The circles intersect at X and Y. Given that $PQ = 9$ cm, find the area of triangle PXQ.

6 Sanjay is making a kite.

The diagram shows a sketch of his kite.

Calculate the area of the material required to make the kite.

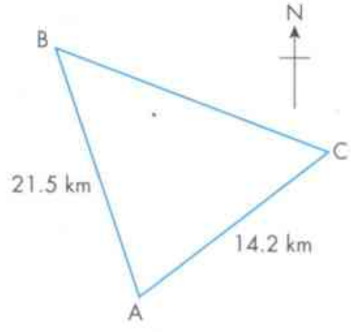


7 The triangular area ABC is to be made a national park.

The bearing of B from A is 324°.

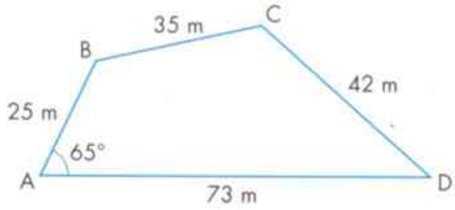
The bearing of C from A is 42°.

Calculate the area of the park.



EXTENDED

8 The diagram shows the dimensions of a four-sided field.

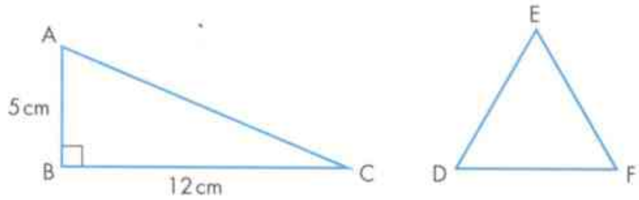


a Show that the length of the diagonal BD is 66 metres to the nearest metre.

b Calculate the size of angle C.

c Calculate the area of the field.

9



Triangle ABC is right-angled.

Triangle DEF is isosceles.

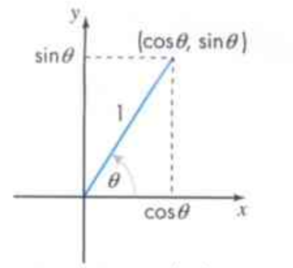
They have the same perimeter.

Calculate the area of triangle DEF.

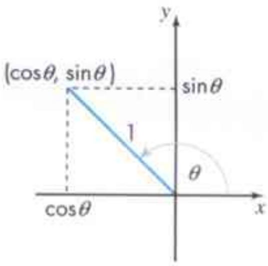
26.11 Sine, cosine and tangent of any angle

E

In Section 26.8 you used a rod of length 1 on a coordinate grid to find the sine and cosine of obtuse angles.

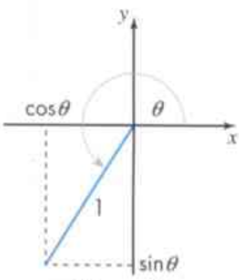


cosθ and sinθ are both positive

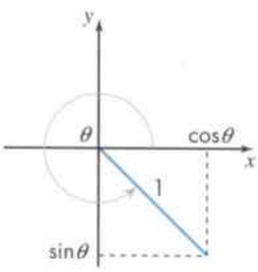


cosθ is negative
sinθ is positive

You can use the same idea to find the sine and cosine of a reflex angle. That is an angle bigger than 180°.

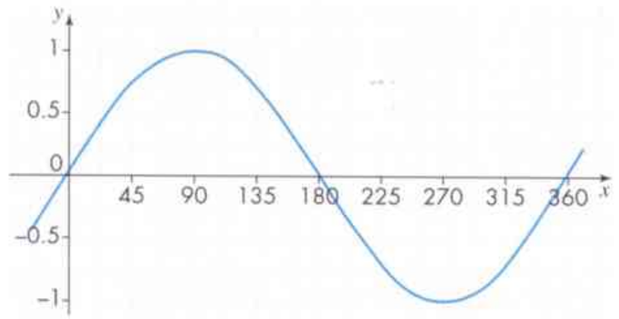


cosθ is negative
sinθ is positive

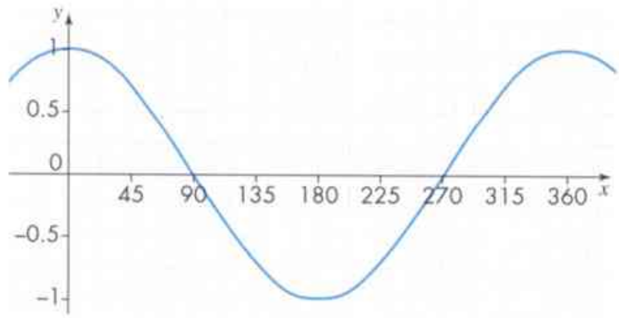


cosθ is positive
sinθ is negative

These graphs show the sine and cosine of any angle between 0° and 360°

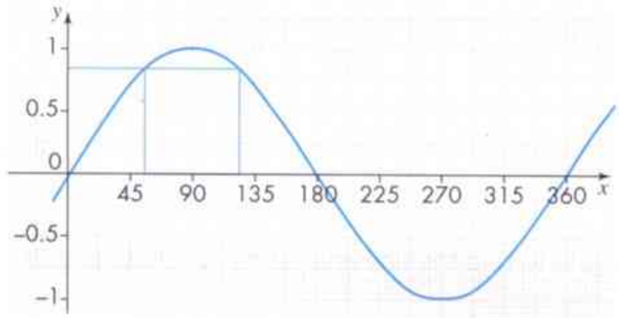


$y = \sin x°$



$y = \cos x°$

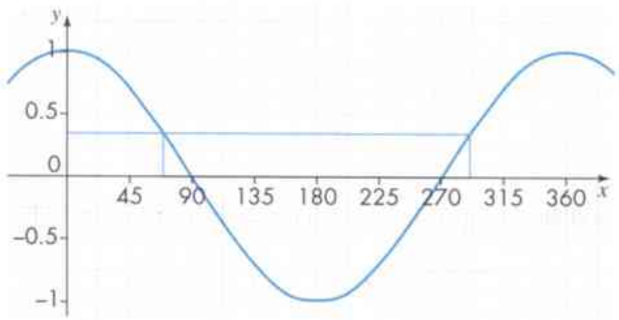
Notice the symmetry of the graphs.



$y = \sin x°$

180 – 55 = 125 and so 55° and 125° have the same sine.

sin 55° = sin 125° = 0.819 to 3 d.p.



$y = \cos x°$

360 – 70 = 290 and so 70° and 290° have the same cosine.

cos 70° = cos 290° = 0.342 to 3 d.p.

Here are some more examples of the symmetry of the sine graph:

- 10° and 180 – 10 = 170° have the same sine.
- 50° and 180 – 50 = 130° have the same sine.
- 205° and 360 – 15 = 345° have the same sine.
- sin 30° = 0.5 and sin (180 + 30)° = sin 210° = –0.5

Here are some more examples of the symmetry of the cosine graph:

- 10° and 360 – 10 = 350° have the same cosine.
- 125° and 360 – 125 = 235° have the same cosine.
- cos 70° = 0.342 and cos (180 – 70)° = cos 110° = –0.342

You need to remember the shapes of these graphs and their symmetries.

Example 23

Solve these equations, giving your answers to the nearest degree.

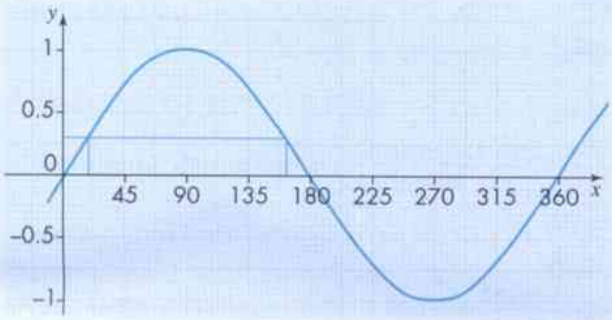
a $\sin x = 0.3$ $\quad 0° \leqslant x \leqslant 360°$

b $\sin w = 0.3$ $\quad 0° \leqslant w \leqslant 360°$

c $2 \cos y + 1 = 0$ $\quad 0° \leqslant y \leqslant 360°$

a A calculator gives $\sin^{-1} 0.3 = 17°$. This is one answer.

Look at the symmetry of the sine graph.

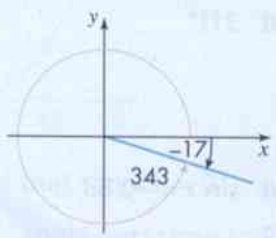


Another answer is 180 – 17 = 163°

There are two possible answers, x = 17° or 163°

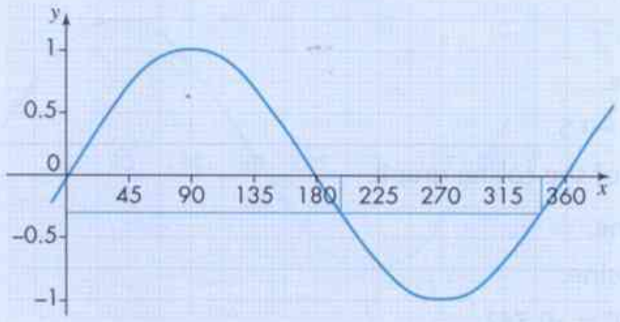
b A calculator gives $\sin^{-1} (-0.3) = -17°$ but this is outside the interval $0° \leqslant w \leqslant 360°$

–17° is a *clockwise* turn of 17°. It is the same as an *anticlockwise* turn of 360 – 17 = 343°



One solution is 343°.

To find another, look at the symmetry of the sine graph.



Another solution is 180 + 17 = 197°.

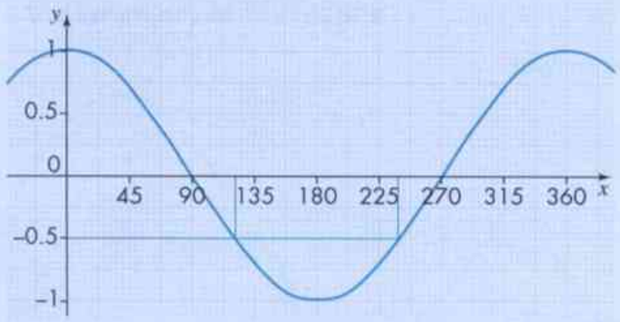
There are two possible answers, w = 197° or 343°

c Rearrange the equation $\quad 2\cos y = -1$

Divide by 2 $\quad \cos y = -0.5$

A calculator gives $\cos^{-1}(-0.5) = 120°$ and this is one answer.

Look at the symmetry of the cosine graph.



Another answer is 360 − 120 = 240°

There are two possible answers, y = 120° or 240°

EXERCISE 26P

EXTENDED

1 Find an angle between 0° and 360° that has the same sine as:

a 80° b 146° c 215° d 306°

2 Find an angle between 0° and 360° that has the same cosine as:

a 10° b 125° c 208° d 311°

3 Solve these equations when $0° \leqslant x \leqslant 360°$

Give your answers to the nearest degree.

a $\sin x = 0.45$ b $\sin x = 0.83$ c $\sin x = -0.45$ d $\sin x = -0.83$

4 Solve these equations when $0° \leqslant x \leqslant 360°$

Give your answers to the nearest degree.

a $\cos x = 0.8$ **b** $\cos x = -0.23$ **c** $\cos x = -0.92$ **d** $\cos x = 0.087$

5 Solve these equations when $0° \leqslant x \leqslant 360°$

a $\sin x = 0.5$ **b** $\sin x = \frac{\sqrt{3}}{2}$ **c** $\sin x = -\frac{1}{\sqrt{2}}$ **d** $\sin x = -1$

6 Solve these equations when $0° \leqslant x \leqslant 360°$

a $\cos x = -0.5$ **b** $\cos x = \frac{\sqrt{3}}{2}$ **c** $\cos x = \frac{1}{\sqrt{2}}$ **d** $\cos x = 0$

7 Solve these equations when $0° \leqslant x \leqslant 360°$

Round your answers to 1 d.p.

a $\cos x = 0.05$ **b** $\sin x = 0.812$ **c** $\cos x + 0.65 = 0$ **d** $\sin x + 0.9 = 0.3$

8 Solve these equations when $0° \leqslant x \leqslant 360°$

Round your answers to 1 d.p.

a $3 \sin x = 2$ **b** $5 \cos x - 4 = 0$ **c** $7 \sin x + 5 = 0$

9 Two different obtuse angles have the same sine. Find the sum of the two angles.

10 Solve the equation $4(\sin x)^2 = 1$ $0° \leqslant x \leqslant 360°$

11 Solve the equation $2(\cos x)^2 = 1$ $0° \leqslant x \leqslant 360°$

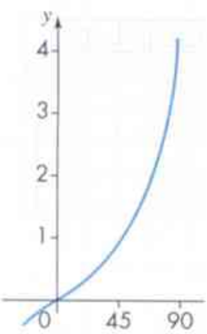
The tangent of any angle

In a right-angled triangle $\tan \theta = \frac{\text{Opposite side}}{\text{Adjacent side}}$

You know that in a right-angled triangle $\tan \theta = \frac{\text{Opposite side}}{\text{Adjacent side}}$

You can see from this diagram that $\tan \theta = \frac{\sin \theta}{\cos \theta}$

Here is a graph of $y = \tan x$ where $0° \leqslant x \leqslant 90°$



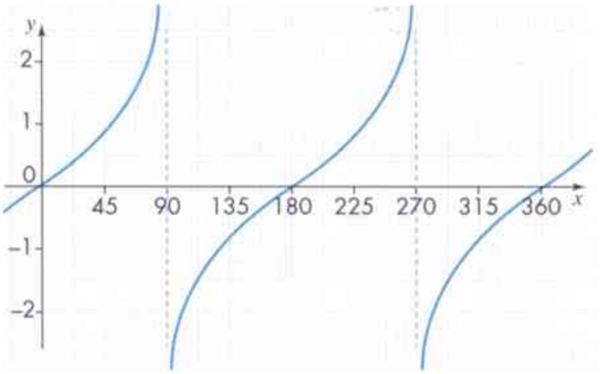
Notice that tan 45° = 0

As the angle gets close to 90° the tangent becomes very large.

tan 90° is not defined because cos 90° = 0

You can use $\tan \theta = \frac{\sin \theta}{\cos \theta}$ to define the tangent of any angle, not just acute ones.

Here is a graph of $y = \tan x$ where $0° \leqslant x \leqslant 180°$



The graph is in three separate branches.

The lines $x = 90$ and $x = 270$ are asymptotes. The graph gets closer and closer to those lines as y gets larger.

The section from 180° to 360° is identical to the section from 0° to 180°

This means that tan 10° = tan 190° and tan 20° = tan 200° and so on.

Example 24

Solve these equations for $0° \leqslant x \leqslant 360°$. Give your answers to 1 d.p.

a $\tan x = 2$ **b** $4 \tan x + 3 = 0$

a A calculator gives $\tan^{-1} 2 = 63.4$ so this is one answer.

From the symmetry of the graph, another answer is 180 + 63.4 = 243.4°

The two answers are $x = 63.4°$ and 243.4°

b Rearrange as $\tan x = -0.75$

A calculator gives $\tan^{-1}(-0.75) = -36.9°$ but this is outside the interval $0° \leqslant x \leqslant 360°$.

However −36.9° is a clockwise turn of 36.9°

It is the same as an anticlockwise turn of 360 − 36.9 = 323.1°

One answer is 323.1°

By the symmetry of the tangent graph, another answer is 323.1 − 180 = 143.1°

The two answers are $x = 143.1°$ and 323.1°

EXERCISE 26Q

EXTENDED

1 Find the angles between 0° and 360° that have the same tangent as

a 35° **b** 85° **c** 118° **d** 200° **e** 243° **f** 337°

2 Solve these equations when $0° \leqslant x \leqslant 360°$.

a $\tan x = 1$ **b** $\tan x = -1$ **c** $\tan x = \sqrt{3}$ **d** $\tan x = -\sqrt{3}$

3 Solve these equations when $0° \leqslant x \leqslant 360°$. Give your answers to 1 d.p.

a $\tan x = 0.2$ b $\tan x = 5$ c $\tan x = -0.35$ d $\tan x = -4.17$

4 The sine of an angle is 0.6 and the cosine of the angle is −0.4

Find the tangent of the angle.

5 Solve these equations when $0° \leqslant x \leqslant 360°$. Give your answers to 1 d.p.

a $8 \tan x = 3$ b $3 \tan x = 8$ c $25 \tan x + 18 = 0$ d $9 \tan x + 40 = 0$

6 Solve these equations $(\tan x)^2 = 1$ and $0° \leqslant x \leqslant 360°$.

a $(\tan x)^2 = 1$ b $(\tan x)^2 = 3$ c $3(\tan x)^2 = 1$

7 Solve the equation $(\tan x)^2 + \tan x - 12 = 0$ and $0° \leqslant x \leqslant 360°$.

8 The difference between two angles is 180°. What can you say about

a the tangents of the angles?

b the sines of the angles?

c the cosines of the angles?

Check your progress

Core

- I can use Pythagoras' theorem the calculate a side of a right-angled triangle
- I can use sine, cosine and tangent ratios to calculate a side or an angle of a right-angled triangle

Extended

- I can solve trigonometric problems in two dimensions involving angles of elevation or depression
- I know that the perpendicular distance from a point to a line is the shortest distance to the line
- I can extend the sine, cosine and tangent values to angles between 90° and 360°
- I know the shapes and properties of graphs of trigonometric functions
- I can find the shortest distance from a point to a line
- I can solve simple trigonometric equations
- I can solve problems using the sine rule or the cosine rule for any triangle
- I can use the formula area of a triangle = $\frac{1}{2}ab \sin C$
- I can solve simple trigonometric problems in three dimensions, including the angle between a line and a plane

Chapter 27

Mensuration

Topics	Level	Key words
1 Perimeter and area of a rectangle	CORE	length, width, perimeter, area
2 Area of a triangle	CORE	base, perpendicular height
3 Area of a parallelogram	CORE	parallelogram
4 Area of a trapezium	CORE	trapezium
5 Circumference and area of a circle	CORE	circumference, diameter, radius, π
6 Surface area and volume of a cuboid	CORE	volume, cuboid, surface area, litre
7 Volume and surface area of a prism	CORE	cross-section, prism
8 Volume and surface area of a cylinder	CORE	curved surface, cylinder
9 Sectors and arcs: 1	CORE	arc, sector, subtended
10 Sectors and arcs: 2	EXTENDED	
11 Volume of a pyramid	CORE	pyramid, vertical height, vertex
12 Volume and surface area of a cone	CORE	cone, vertical height, slant height
13 Volume and surface area of a sphere	CORE	sphere

In this chapter you will learn how to:

CORE	EXTENDED
Carry out the following calculations involving multiples of π where appropriate: • the perimeter and area of a rectangle, triangle, parallelogram and trapezium and compound shapes derived from these (C5.2 and E5.2) • the circumference and area of a circle (C5.3 and E5.3) • the volume of a cuboid, prism and cylinder (C5.4 and E5.4) • the surface area of a cuboid, prism and a cylinder (C5.4 and E5.4) • the areas and volumes of compound shapes (C5.5 and E5.5) • the arc length and sector area as fractions of the circumference and area of a circle (C5.3 and E5.3) • the surface area and volume of a sphere, pyramid and cone (given formulae for the sphere, pyramid and cone). (C5.4 and E5.4)	**Solve problems involving:** • the arc length and sector area with more complicated angle fractions. (E5.3)

Why this chapter matters

People have always needed to measure areas and volumes.

In everyday life, you will, for instance, need to find the area to work out how many tiles to buy to cover a floor; or you will need to find the volume to see how much water is needed to fill a swimming pool. You can do this quickly using formulae.

Measuring the world

From earliest times, farmers have wanted to know the area of their fields to see how many crops they could grow or animals they could support. When land is bought and sold, the cost depends on the area.

Volumes are important too. Volumes tell you how much space there is inside any structure. Whether it is a house, barn, aeroplane, car or office, the volume is important. In some countries there are regulations about the number of people who can use an office, based on the volume of the room.

Volumes of containers for liquids also need to be measured. Think, for example, of a car fuel tank, the water tank in a building, or a reservoir. It is important to be able to calculate the capacity of all these things.

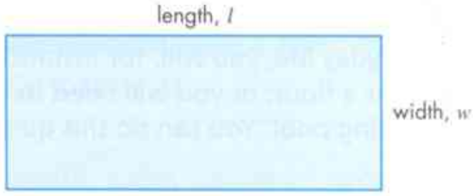
So how do you measure areas and volumes? In this chapter, you will learn formulae that can be used to calculate areas and volumes of different shapes, based on a few measurements.

Many of these formulae were first worked out thousands of years ago. They are still in use today because they are important in everyday life.

The process of calculating areas and volumes using formulae is called **mensuration**.

27.1 Perimeter and area of a rectangle

The **perimeter** of a rectangle is the total distance around the outside.



perimeter $= l + w + l + w$

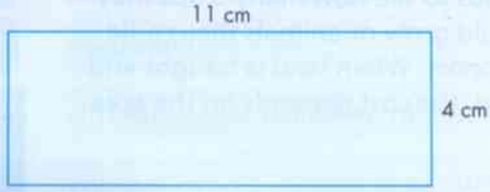
$= 2(l + w)$

The **area** of a rectangle is **length × width**.

area $= lw$

Example 1

Calculate the area and perimeter of this rectangle.



Area of rectangle = length × width
= 11 cm × 4 cm
= 44 cm^2

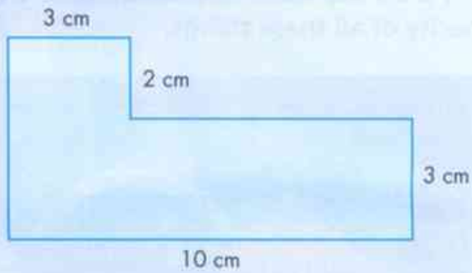
Perimeter = 2 × 11 + 2 × 4
= 30 cm

Some two-dimensional shapes are made up of two or more rectangles.

These shapes can be split into simpler shapes, which makes it easy to calculate their areas.

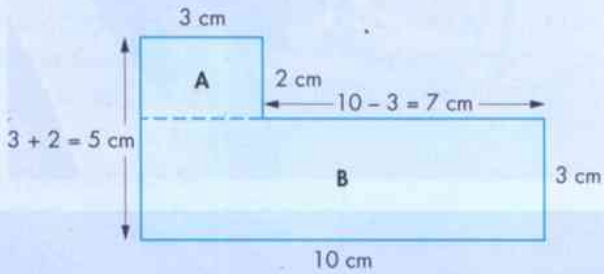
Example 2

Find the area and perimeter of the shape shown on the right.



First, split the shape into two rectangles, A and B and find the missing lengths.

The perimeter is
3 + 2 + 7 + 3 + 10 + 5 = 30 cm
area of A = 2 × 3 = 6 cm^2
area of B = 10 × 3 = 30 cm^2

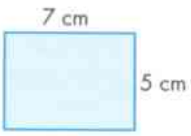


The area of the shape is:
area of A + area of B = 6 + 30
= 36 cm^2

EXERCISE 27A

1 Calculate the area and the perimeter for each of these rectangles.

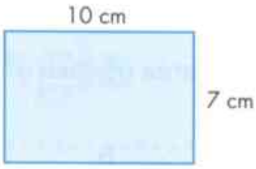
a



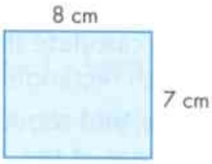
b

c

d



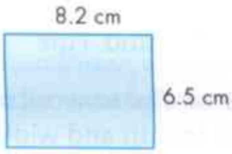
e



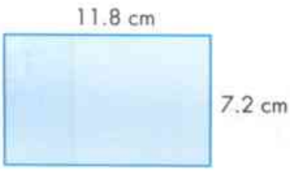
f

2 Calculate the area and the perimeter for each of these rectangles.

a



b



3 A rectangular field is 150 m long and 45 m wide.

Fencing is needed to go all the way around the field.

The fencing is sold in 10-metre long pieces.

How many pieces are needed?

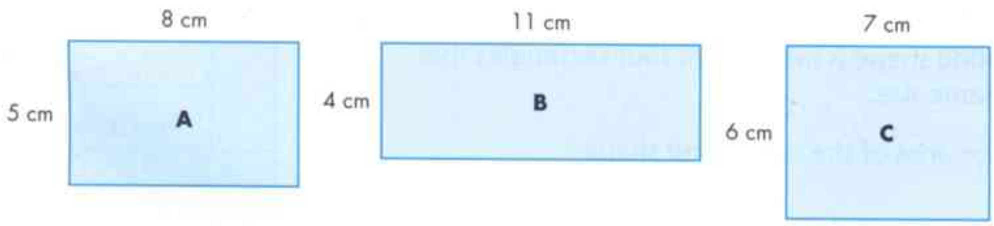
4 A soccer pitch is 160 m long and 70 m wide.

a Before a game, the players have to run about 1500 m to help them loosen up. How many times will they need to run round the perimeter of the pitch to do this?

b The groundsman waters the pitch at the rate of 100 m^2 per minute. How long will it take him to water the whole pitch?

5 What is the perimeter of a square with an area of 100 cm^2?

6 Which rectangle has the largest area? Which has the largest perimeter?



Explain your answers.

CORE

7 Doubling the length and width of a rectangle doubles the area of the rectangle.

Is this statement:

- always true
- sometimes true
- never true?

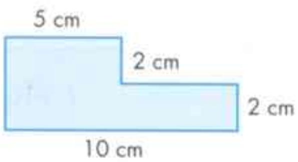
Explain your answer.

Advice and Tips

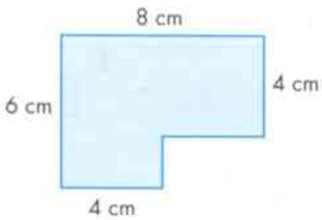
Draw some diagrams with different lengths and widths.

8 Calculate the perimeter and area of each of these compound shapes below.

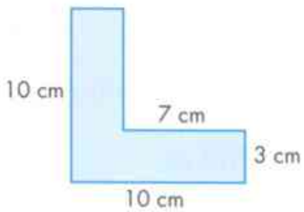
a



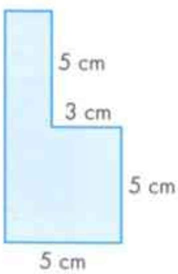
b



c



d



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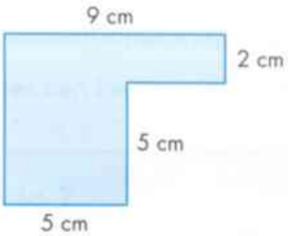

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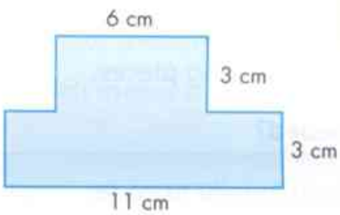
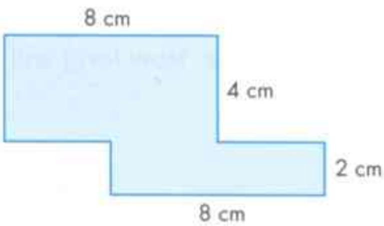
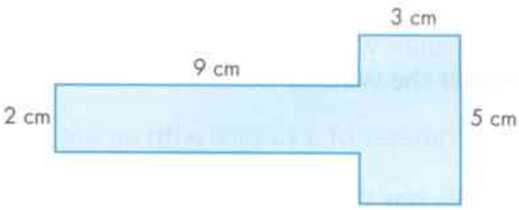


g



h



Advice and Tips

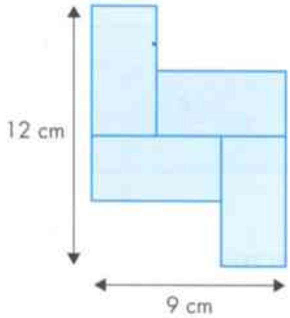
- First, split the compound shape into rectangles.
- Then, calculate the area of each rectangle
- Finally, add together the areas of the rectangles.

Advice and Tips

Be careful to work out the length and width of each separate rectangle. You will usually have to add or subtract lengths to find some of these.

9 This compound shape is made from four rectangles that are all the same size.

Work out the area of the compound shape.



10 This shape is made from five squares that are all the same size.

It has an area of 80 cm^2.

Work out the perimeter of the shape.

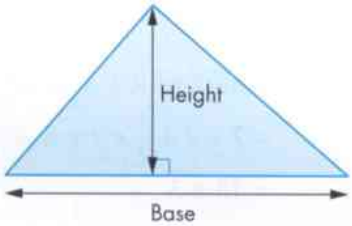
27.2 Area of a triangle

The area of any triangle is given by the formula:

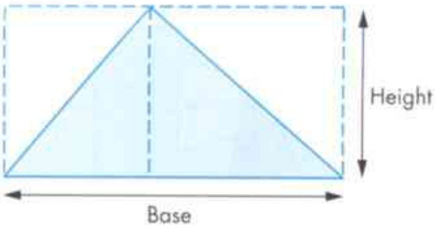
area = $\frac{1}{2}$ × **base** × **perpendicular height**

As an algebraic formula, this is written as:

$A = \frac{1}{2}bh$

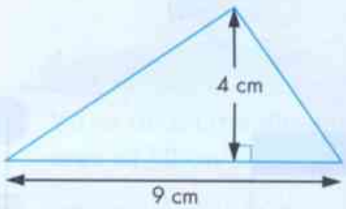


This diagram shows why the area for a triangle is half the area of a rectangle with the same base and height.



Example 3

Calculate the area of this triangle.



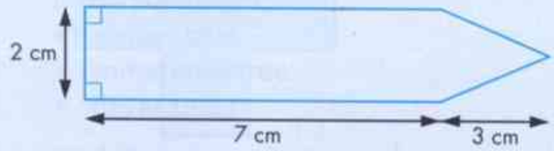
Area = $\frac{1}{2}$ × 9 cm × 4 cm

= $\frac{1}{2}$ × 36 cm^2

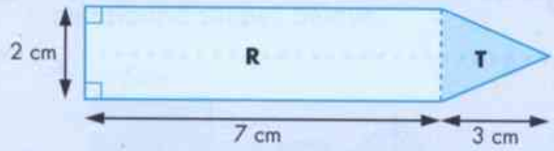
= 18 cm^2

Example 4

Calculate the area of the shape shown below.



This shape can be split into a rectangle (R) and a triangle (T).



Area of the shape = area of R + area of T

$= 7 \times 2 + \frac{1}{2} \times 2 \times 3$

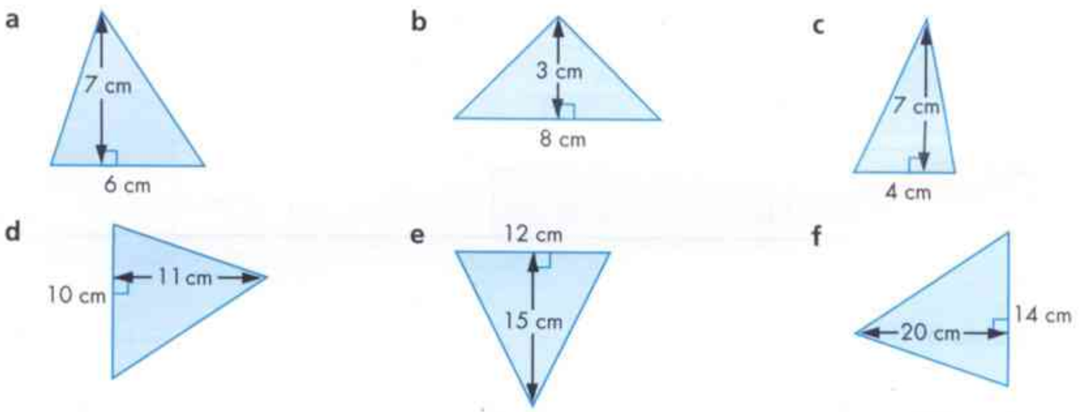
$= 14 + 3$

$= 17 \text{ cm}^2$

EXERCISE 27B

CORE

1 Calculate the area of each of these triangles.

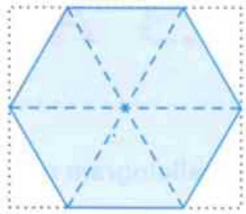


2 Copy and complete the table for triangles **a** to **f**.

	Base	Perpendicular height	Area
a	8 cm	7 cm	
b		9 cm	36 cm^2
c		5 cm	10 cm^2
d	4 cm		6 cm^2
e	6 cm		21 cm^2
f	8 cm	11 cm	

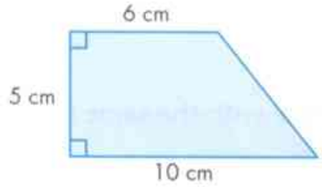
3 This regular hexagon has an area of 48 cm^2.

What is the area of the square that surrounds the hexagon?

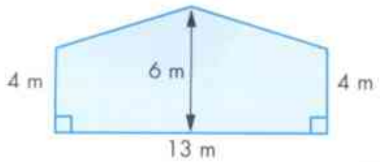


4 Find the area of each of these shapes.

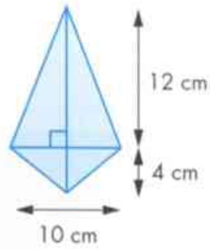
a



b



c

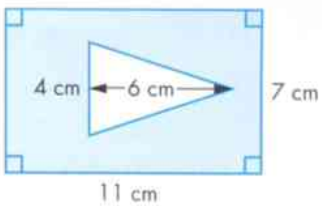


Advice and Tips

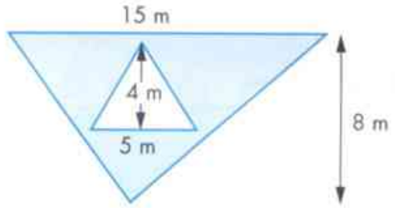
Refer to Example 4 on how to find the area of a compound shape.

5 Find the area of each shaded shape.

a



b



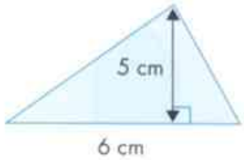
Advice and Tips

Find the area of the outer shape and subtract the area of the inner shape.

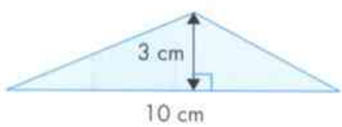
6 Write down the dimensions of two different-sized triangles that have the same area of 50 cm^2.

7 Which triangle is the odd one out? Give a reason for your answer.

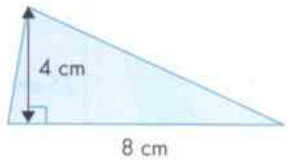
a



b

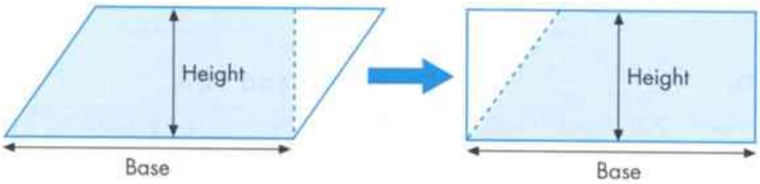


c



27.3 Area of a parallelogram

A **parallelogram** can be changed into a rectangle by moving a triangle.



This shows that the area of the parallelogram is the area of a rectangle with the same base and height. The formula is:

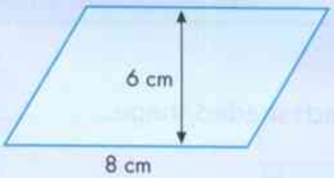
area of a parallelogram = base × height

As an algebraic formula, this is written as:

$A = bh$

Example 5

Find the area of this parallelogram.



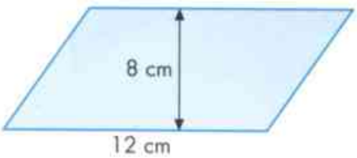
Area = 8 cm × 6 cm
= 48 cm^2

EXERCISE 27C

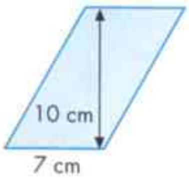
CORE

1 Calculate the area of each parallelogram below.

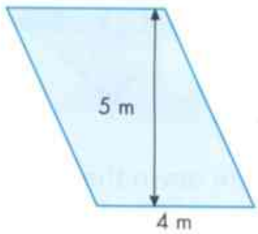
a



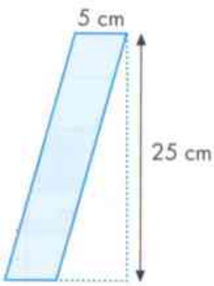
b



c



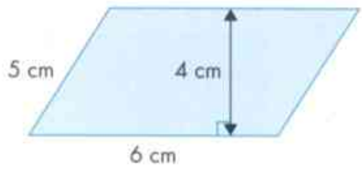
d



e

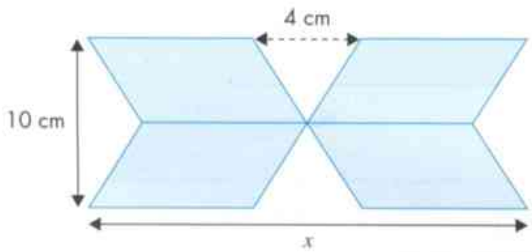
f

2 Sandeep says that the area of this parallelogram is 30 cm^2.



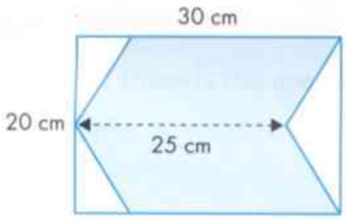
Is she correct? Give a reason for your answer.

3 This shape is made from four parallelograms that are all the same size. The area of the shape is 120 cm^2.



Work out the length marked x on the diagram.

4 This logo, made from two identical parallelograms, is cut from a sheet of card.

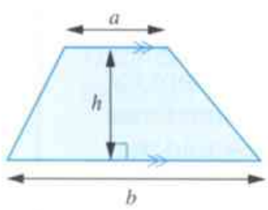


a Calculate the area of the logo.

b How many logos can be cut from a sheet of card that measures 1 m by 1 m?

27.4 Area of a trapezium

You can calculate the area of a **trapezium** by finding the average of the lengths of its parallel sides and multiplying this by the perpendicular height between them.

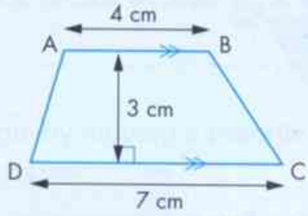


The area of a trapezium is given by this formula:

$$A = \frac{1}{2}(a + b)h$$

Example 6

Find the area of the trapezium ABCD.



Area = $\frac{1}{2}(4 + 7) \times 3$

$= \frac{1}{2} \times 11 \times 3$

$= 16.5\ cm^2$

EXERCISE 27D

CORE

1 Copy and complete the table for the trapezia a–g.

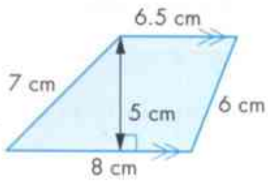
	Parallel side 1	Parallel side 2	Perpendicular height	Area
a	8 cm	4 cm	5 cm	
b	10 cm	12 cm	7 cm	
c	7 cm	5 cm	4 cm	
d	5 cm	9 cm	6 cm	
e	3 cm	13 cm	5 cm	
f	4 cm	10 cm		$42\ cm^2$
g	7 cm	8 cm		$22.5\ cm^2$

Advice and Tips

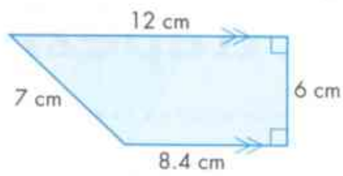
Trapezia is the plural of trapezium.

2 Calculate the perimeter and the area of each trapezium.

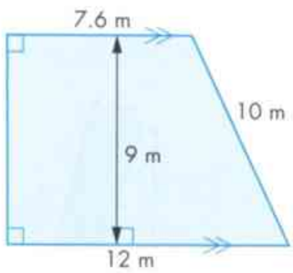
a



b



c

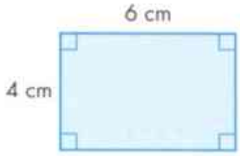


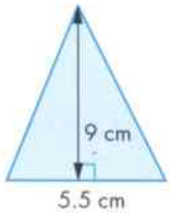
Advice and Tips

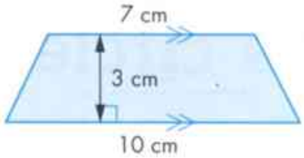
Make sure you use the right measurement for the height. Sometimes you might be told the slant side length, which is not used for the area.

3 A trapezium has an area of $25\ cm^2$. Its vertical height is 5 cm. Work out a possible pair of lengths for the two parallel sides.

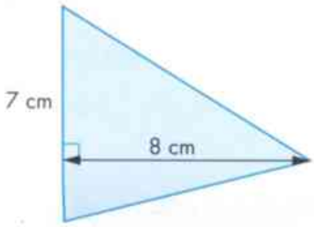
4 Which of these shapes has the largest area?

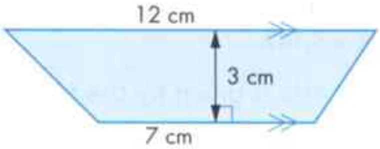
a


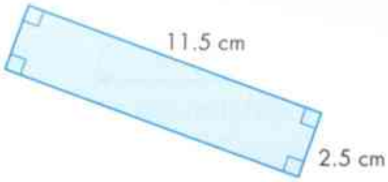
b


c


5 Which of these shapes has the smallest area?

a


b


c


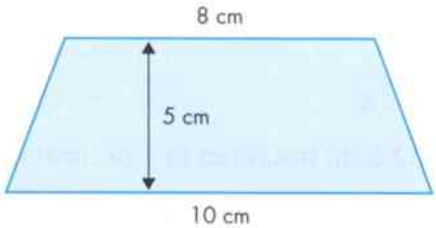
6 Which of these is the area of this trapezium?

a 45 cm^2

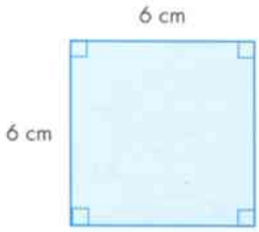
b 65 cm^2

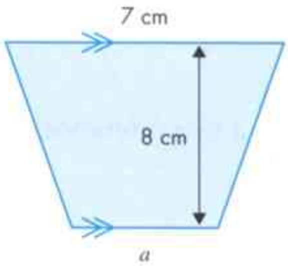
c 70 cm^2

You must show your workings.

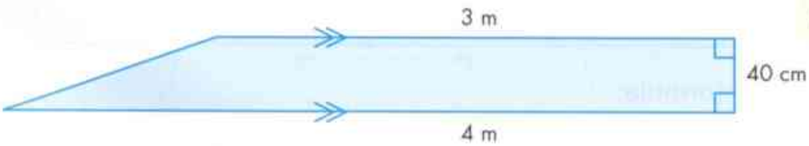


7 Work out the value of *a* so that the square and the trapezium have the same area.





8 The side of a ramp is a trapezium, as shown in the diagram. Calculate its area, giving your answer in square metres.



Advice and Tips

Change the height into metres first.

27.5 Circumference and area of a circle

The perimeter of a circle is called the **circumference**.

You can calculate the circumference, C, of a circle by multiplying its **diameter**, d, by π.

The value of π is found on all scientific calculators, with $\pi = 3.141\,592\,654$, but if it is not on your calculator, then take $\pi = 3.142$.

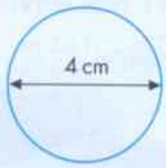
The circumference of a circle is given by the formula:

circumference = π × diameter *or* $C = \pi d$

As the diameter is twice the **radius**, r, this formula can also be written as $C = 2\pi r$.

Example 7

Calculate the circumference of the circle with a diameter of 4 cm.



Use the formula:

$C = \pi d$

$= \pi \times 4$

$= 12.6$ cm (rounded to 1 decimal place)

Remember The length of the radius of a circle is half the length of its diameter. So, when you are given a radius, in order to find a circumference you must first double the radius to get the diameter.

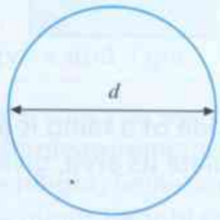
Example 8

Calculate the diameter of a circle that has a circumference of 40 cm.

$C = \pi \times d$

$40 = \pi \times d$

$d = \frac{40}{\pi} = 12.7$ cm (rounded to 1 decimal place)



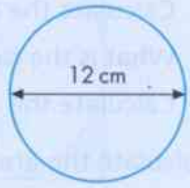
The area, A, of a circle is given by the formula:

area = π × radius^2 *or* $A = \pi \times r \times r$ *or* $A = \pi r^2$

Remember This formula uses the radius of a circle. So, when you are given the diameter of a circle, you must *halve* it to get the radius.

Example 9

Calculate the area of a circle with a diameter of 12 cm. Give your answer as a multiple of π.



First, halve the diameter to find the radius:

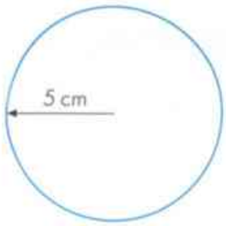
radius = 12 ÷ 2 = 6 cm

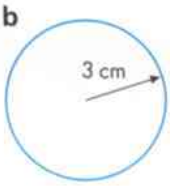
Then, find the area:

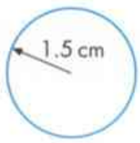
$$\text{area} = \pi r^2$$
$$= \pi \times 6^2$$
$$= 36\pi \text{ cm}^2$$

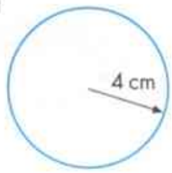
EXERCISE 27E

1 Calculate the circumference and area of each circle. Give your answers as a multiple of π.

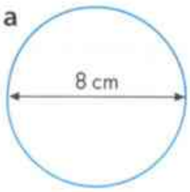
a 

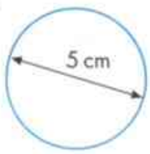
b 

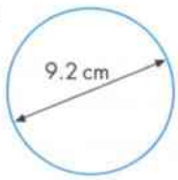
c 

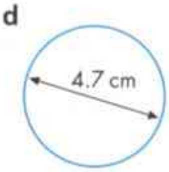
d 

2 Calculate the circumference and area of each circle. Give your answers to 1 decimal place.

a 

b 

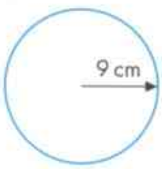
c 

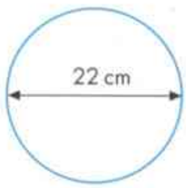
d 

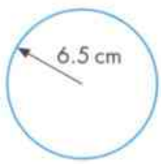
3 Calculate:

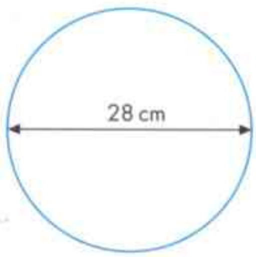
i the circumference

ii the area of each of these circles. Give your answers as a multiple of π and also to 1 decimal place.

a 

b 

c 

d 

CORE

4 A circle has a circumference of 60 cm.

a Calculate the diameter of the circle to 1 decimal place.

b What is the radius of the circle to 1 decimal place?

c Calculate the area of the circle to 1 decimal place.

5 Calculate the area of a circle with a circumference of 110 cm.

6 The circumference of a circle is 40π cm.

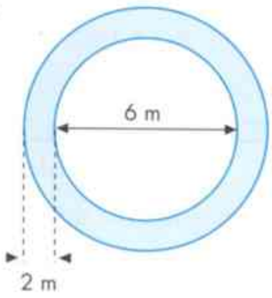
Find **a** the radius of the circle **b** the area of the circle

7 Calculate the area of the shaded part of each of these diagrams. Give your answer as a multiple of π.

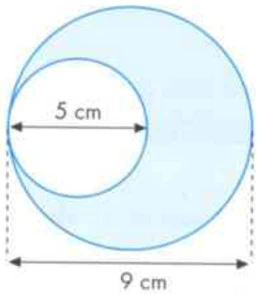
Advice and Tips

In each diagram, subtract the area of the small circle from the area of the large circle.

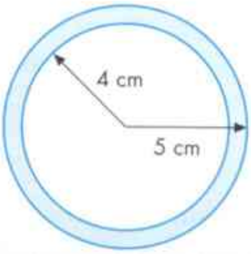
a



b

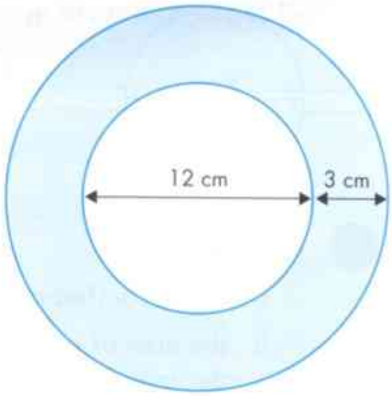


c



8 The diagram shows a circular photograph frame.

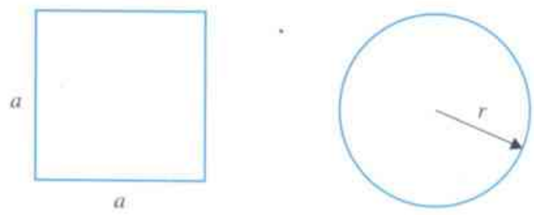
Work out the area of the frame. Give your answer as a multiple of π.



9 A square has sides of length a and a circle has radius r.

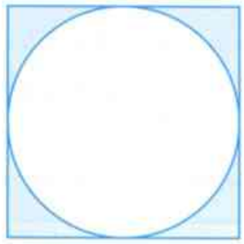
The area of the square is equal to the area of the circle.

Show that $r = \frac{a}{\sqrt{\pi}}$



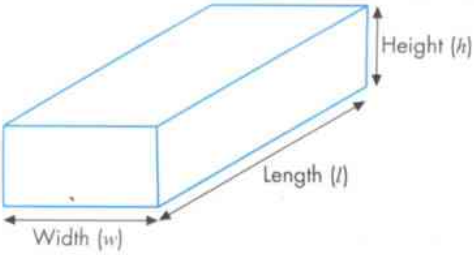
10 A circle fits exactly inside a square of sides 10 cm.

Calculate the area of the shaded region. Give your answer to 1 decimal place.



27.6 Surface area and volume of a cuboid

A cuboid is a box shape, all six faces of which are rectangles.



Every day you will see many examples of **cuboids**, such as food packets, smart phone – and even this book.

The **volume** of a cuboid is given by the formula:

volume = length × width × height *or* $V = l \times w \times h$ *or* $V = lwh$

You can claculate the **surface area** of a cuboid by finding the total area of the six faces, which are rectangles. Notice that each pair of opposite rectangles have the same area. So, from the diagram above:

area of top and bottom rectangles = 2 × length × width = $2lw$

area of front and back rectangles = 2 × height × width = $2hw$

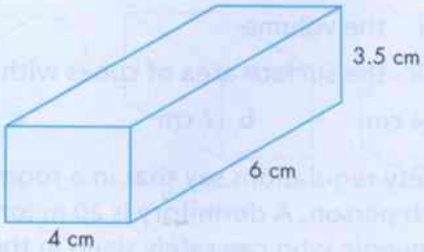
area of two side rectangles = 2 × height × length = $2hl$

Hence, the surface area of a cuboid is given by the formula:

surface area = $A = 2lw + 2hw + 2hl$

Example 10

Calculate the volume and surface area of this cuboid.



Volume = $V = lwh = 6 \times 4 \times 3.5 = 84\ \text{cm}^3$

Surface area = $A = 2lw + 2hw + 2hl$

$= (2 \times 6 \times 4) + (2 \times 3.5 \times 4) + (2 \times 3.5 \times 6)$

$= 48 + 28 + 42 = 118\ \text{cm}^2$

Note:

$1\ \text{cm}^3 = 1000\ \text{mm}^3$ and $1\ \text{m}^3 = 1\,000\,000\ \text{cm}^3$

$1000\ \text{cm}^3 = 1$ litre

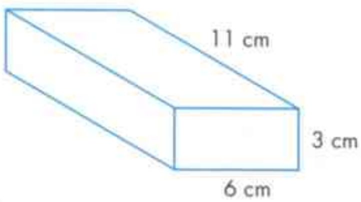
$1\ \text{m}^3 = 1000$ litres

EXERCISE 27F

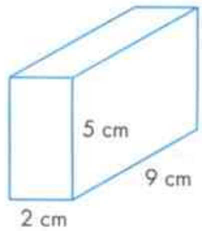
CORE

1 Find **i** the volume and **ii** the surface area of each of these cuboids.

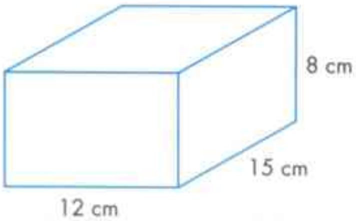
a



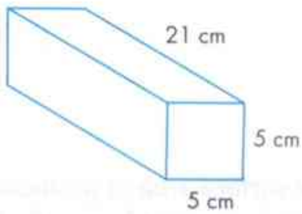
b



c



d



2 Find the capacity of a fish-tank with dimensions: length 40 cm, width 30 cm and height 20 cm. Give your answer in litres.

3 Find the volume of each cuboid

a The area of the base is $40\ \text{cm}^2$ and the height is 4 cm.

b The base has one side 10 cm and the other side 2 cm longer, and the height is 4 cm.

c The area of the top is $25\ \text{cm}^2$ and the depth is 6 cm.

4 Calculate:

i the volume

ii the surface area of cubes with these edge lengths.

a 4 cm **b** 7 cm **c** 10 mm **d** 5 m **e** 12 m

5 Safety regulations say that in a room where people sleep there should be at least $12\ \text{m}^3$ for each person. A dormitory is 20 m long, 13 m wide and 4 m high. What is the greatest number of people who can safely sleep in the dormitory?

6 Copy and complete the table for cuboids **a** to **e**.

	Length	Width	Height	Volume
a	8 cm	5 cm	4.5 cm	
b	12 cm	8 cm		480 cm^3
c	9 cm		5 cm	270 cm^3
d		7 cm	3.5 cm	245 cm^3
e	7.5 cm	5.4 cm	2 cm	

7 A tank contains 32 000 litres of water. The base of the tank measures 6.5 m by 3.1 m. Find the depth of water in the tank. Give your answer to one decimal place.

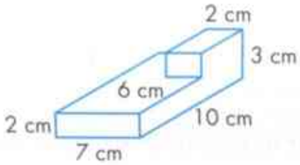
8 A room contains 168 m^3 of air. The height of the room is 3.5 m. What is the area of the floor?

9 What are the dimensions of cubes with these volumes?

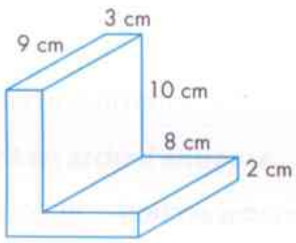
a 27 cm^3 **b** 125 m^3 **c** 8 mm^3 **d** 1.728 m^3

10 Calculate the volume of each of these shapes.

a



b



Advice and Tips

Split the solid into two separate cuboids and work out the dimensions of each of them from the information given.

11 A cuboid has volume of 125 cm^3 and a total surface area of 160 cm^2.

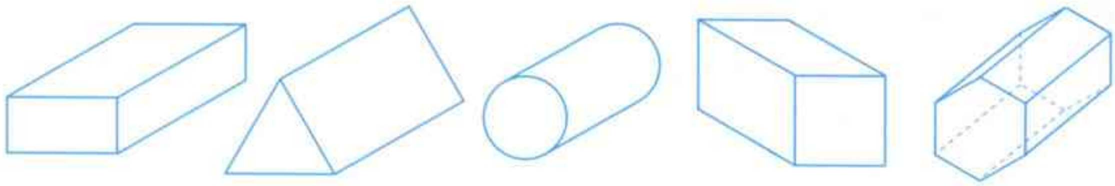
Is it possible that this cuboid is a cube? Give a reason for your answer.

12 The volume of a cube is N cm^3. The area of the cube is N cm^2.

a How long is each side of the cube? **b** What is the value of N?

27.7 Volume and surface area of a prism

A **prism** is a three-dimensional shape that has the same **cross-section** running all the way through it.

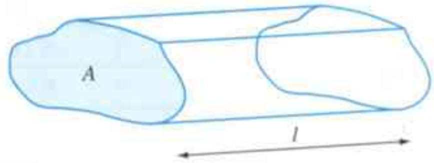


Name:	Cuboid	Triangular prism	Cylinder	Cuboid	Hexagonal prism
Cross-section:	Rectangle	Triangle	Circle	Square	Hexagon

The volume of a prism is found by multiplying the area of its cross-section by the length of the prism (or height if the prism is stood on end).

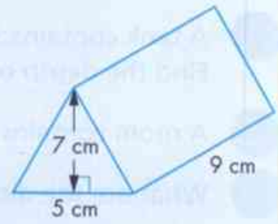
That is, volume of prism = area of cross-section × length

or $V = Al$



Example 11

Find the volume of the triangular prism.



The area of the triangular cross-section $A = \frac{5 \times 7}{2}$

$= 17.5\ \text{cm}^2$

The volume is the area of its cross-section × length = Al

$= 17.5 \times 9$

$= 157.5\ \text{cm}^3$

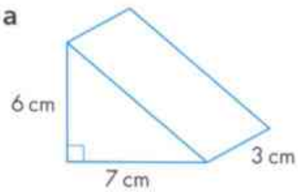
EXERCISE 27G

CORE

1 For each prism shown:

i calculate the area of the cross-section ii calculate the volume.

a



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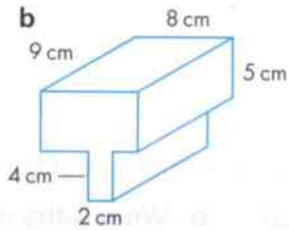

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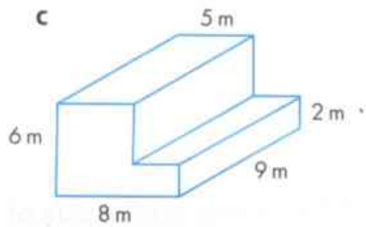
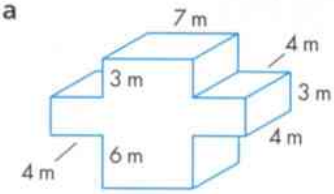


2 Calculate the volume of each of these prisms.

a



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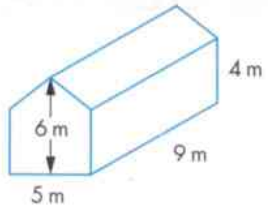

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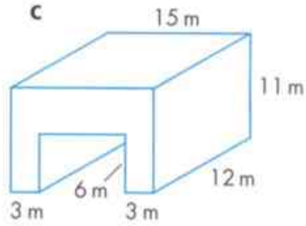
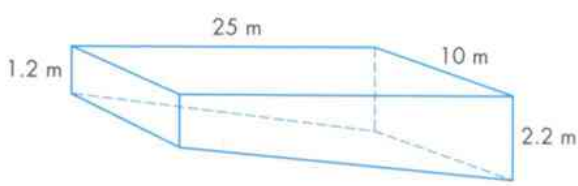


3 A swimming pool is 10 m wide and 25 m long.

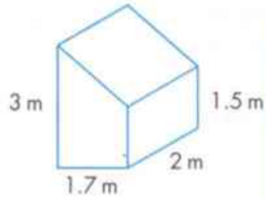
It is 1.2 m deep at one end and 2.2 m deep at the other end. The floor slopes uniformly from one end to the other.



a Explain why the shape of the pool is a prism.

b The pool is filled with water at a rate of 2 m^3 per minute. How long will it take to fill the pool?

4 A building is in the shape of a prism with the dimensions shown in the diagram. Calculate the volume of air (in litres) inside the building.

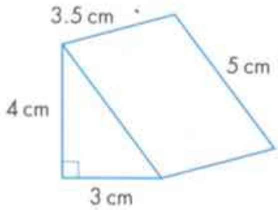


5 Each of these prisms has a uniform cross-section in the shape of a right-angled triangle.

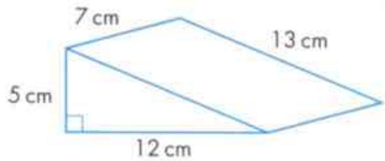
a Find the volume of each prism.

b Find the total surface area of each prism.

i



ii



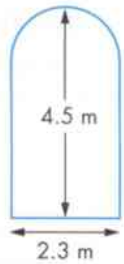
6 The top and bottom of the container shown here are the same size, both consisting of a rectangle, 4 cm by 9 cm, with a semi-circle at each end. The depth is 3 cm. Find the volume of the container.

7 In 2009 the sculptor Anish Kapoor exhibited a work called *Svayambh*. It was a block of red wax in the shape of a prism.

The cross-section was in the shape of an arched entrance.

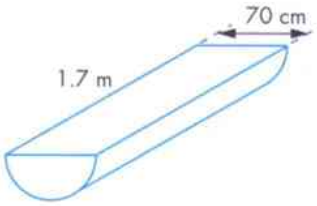
It was 8 m long and weighed 30 tonnes. It slowly travelled through the galleries on a track.

Calculate the volume of wax used.



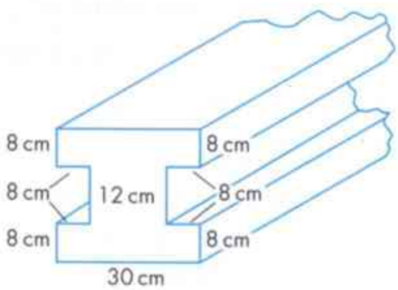
8 A horse trough is in the shape of a semi-circular prism, as shown.

What volume of water will the trough hold when it is filled to the top? Give your answer in litres.

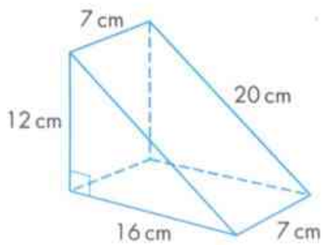


9 The dimensions of the cross-section of a girder (in the shape of a prism), 2 m in length, are shown on the diagram. The girder is made of iron. 1 cm^3 of iron weighs 79 g.

What is the mass of the girder?



10 Calculate the volume of this prism.



27.8 Volume and surface area of a cylinder

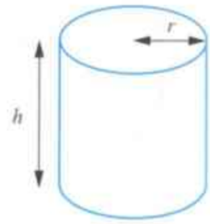
Volume

Since a **cylinder** is an example of a prism, you can calculate its **volume** by multiplying the area of one of its circular ends by the height.

That is,

$$\text{volume} = \pi r^2 h$$

where r is the radius of the cylinder and h is its height or length.



Example 12

What is the volume of a cylinder with a radius of 5 cm and a height of 12 cm?

Volume = area of circular base × height

$= \pi r^2 h = \pi \times 5^2 \times 12 \text{ cm}^3 = 942 \text{ cm}^3$ (3 significant figures)

Surface area

The total surface area of a cylinder is made up of the area of its **curved surface** plus the area of its two circular ends.

The curved surface area, when opened out, is a rectangle with length equal to the circumference of the circular end.

curved surface area = circumference of end × height of cylinder

$= 2\pi rh$ or πdh

area of one end $= \pi r^2$

Therefore, total surface area $= 2\pi rh + 2\pi r^2$ or $\pi dh + 2\pi r^2$

Example 13

What is the total surface area of a cylinder with a radius of 15 cm and a height of 2.5 m?

First, you must change the dimensions to a *common unit*. Use centimetres in this case.

Total surface area $= \pi dh + 2\pi r^2$

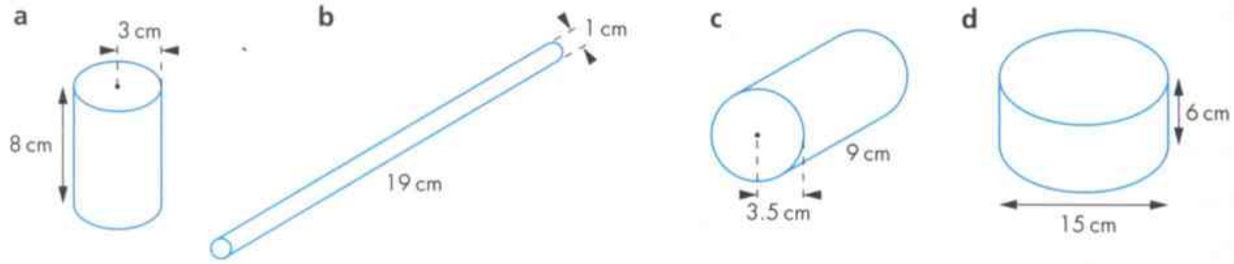
$= \pi \times 30 \times 250 + 2 \times \pi \times 15^2 \text{ cm}^2$

$= 23\,562 + 1414 \text{ cm}^2 = 24\,976 \text{ cm}^2 = 25\,000 \text{ cm}^2$ (3 significant figures)

EXERCISE 27H

1. For the cylinders below find:
   - i the volume
   - ii the total surface area.

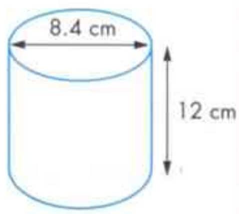
   Give your answers as a multiple of π and also to 3 significant figures.



2. For each of these cylinder dimensions find:
   - i the volume
   - ii the curved surface area.

   Give your answers in terms of π.
   - a Base radius 3 cm and height 8 cm
   - b Base diameter 8 cm and height 7 cm
   - c Base diameter 12 cm and height 5 cm
   - d Base radius of 10 m and length 6 m

3. A solid cylinder has a diameter of 8.4 cm and a height of 12.0 cm. Calculate the volume of the cylinder.



4. A cylindrical food can has a height of 10.5 cm and a diameter of 7.4 cm.

   What can you say about the size of the paper label around the can?

5. A cylindrical container is 65 cm in diameter. Water is poured into the container until it is 1 m deep.

   How much water is in the container? Give your answer in litres.

6. A drinks manufacturer plans a new drink in a can. The quantity in each can must be 330 ml.

   Suggest a suitable height and diameter for the can. (You might like to look at the dimensions of a real drinks can.)

7. Wire is commonly made by putting hot metal through a hole in a plate.

   What length of wire of diameter 1 mm can be made from a 1 cm cube of metal?

8 The engine size of a car is measured in litres. This tells you the total volume of its cylinders. Cylinders with the same volume can be long and thin or short and thick.

In a racing car, the diameter of a cylinder is twice its length. Suggest possible dimensions for a 0.4 litre racing car cylinder.

27.9 Sectors and arcs: 1

A **sector** is part of a circle.

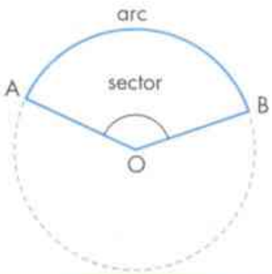
It is formed by two radii and part of the circumference.

The part of the circumference is called an **arc**.

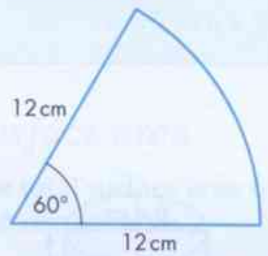
O is the angle of the sector.

The sector is a fraction of the circle.

You can use the angle to find the fraction.



Example 14



This is a sector. Find

a the arc length. **b** the sector area.

Leave π in your answers.

a The sector is a fraction of a circle.

The fraction is $\frac{60}{360} = \frac{1}{6}$

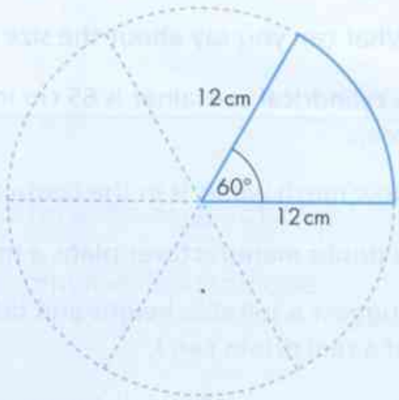
The diameter of the circle is 24 cm.

The circumference of the circle is $\pi \times 24 = 24\pi$ cm

The arc length is $\frac{1}{6}$ of $24\pi = 24\pi \div 6 = 4\pi$ cm

b The area of the circle is $\pi \times 12^2 = 144\pi$ cm²

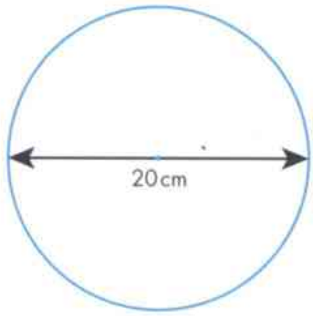
The sector area is $\frac{1}{6}$ of $144\pi = 144\pi \div 6 = 24\pi$ cm²



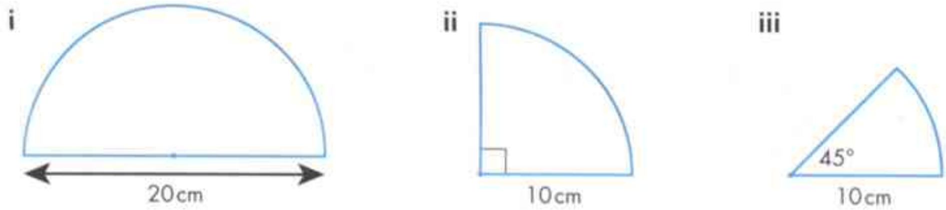
EXERCISE 27I

1 In this question leave π in your answers.

a Find the circumference of this circle.

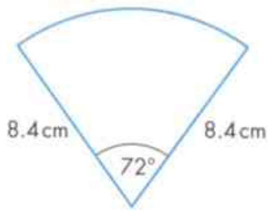


b Find the arc length for each of these sectors.



2 Find the area of each of the shapes in question 1. Leave π in your answers.

3

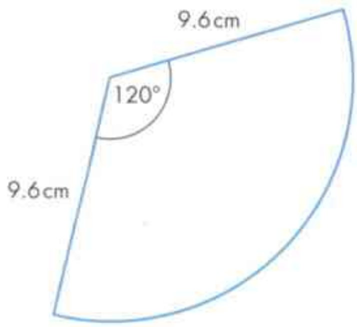


a What fraction of a circle is this sector?

b Find the arc length in cm. Round your answer to 1 d.p.

c Find the area of the sector in cm^2. Round your answer to 1 d.p.

4

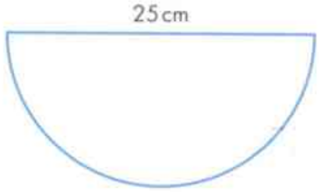


a Find the area of this sector in cm^2. Round your answer to 1 d.p.

b Find the length of the arc in cm. Round your answer to 1 d.p.

c Find the perimeter of the shape.

5

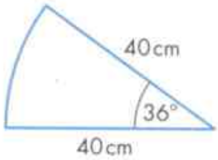


This is a semicircle.

a Find the area.

b Find the perimeter.

6

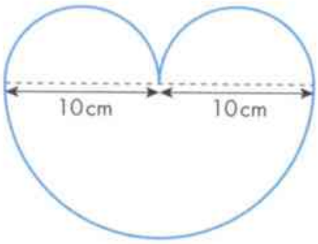


This is a sector.

a Show that the length of the arc is 8π cm.

b Find the area of the sector.

7



This shape is made from 3 semicircles.

a Find the perimeter of the shape. Leave π in your answer.

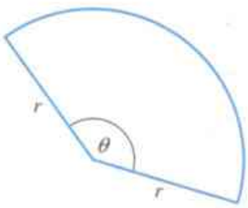
b Find the area of the shape. Leave π in your answer.

27.10 Sectors and arcs: 2

E

In the last section the sectors were all simple fractions of a circle.

You can always use the angle of the sector to find the arc length and sector area, even if it is not a simple fraction.



If the angle of the sector is θ, then:

arc length = $\frac{\theta}{360} \times$ circumference = $\frac{\theta}{360} \times 2\pi r$

and sector area = $\frac{\theta}{360} \times$ circle area = $\frac{\theta}{360} \times \pi r^2$

Example 15

Calculate the arc length and the area of the sector in the diagram.

The sector angle is 28° and the radius is 5 cm. Therefore:

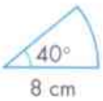
$$\text{arc length} = \frac{28}{360} \times \pi \times 2 \times 5 = 2.4 \text{ cm (1 decimal place)}$$

$$\text{sector area} = \frac{28}{360} \times \pi \times 5^2 = 6.1 \text{ cm}^2 \text{ (1 decimal place)}$$

EXERCISE 27J

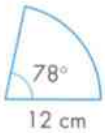
1 For each of these sectors, calculate: **i** the arc length **ii** the sector area.

a

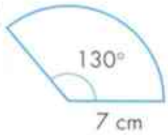


b

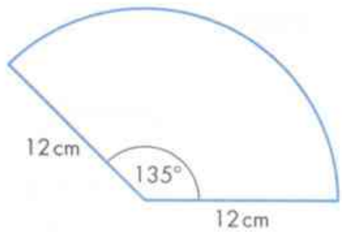
c



d

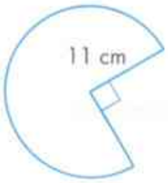


2 For this sector work out: **a** the arc length **b** the sector area Give your answers in terms of π.

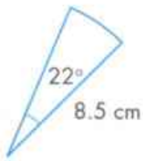


3 Calculate the total perimeter of each of these sectors.

a

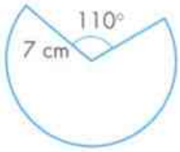


b



4 Calculate the area of each of these sectors.

a



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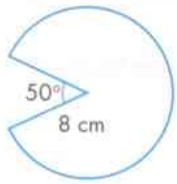
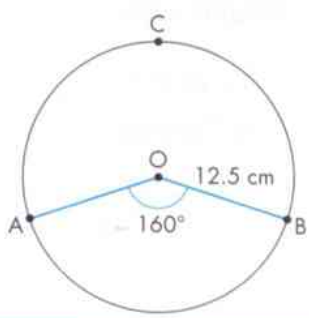


5 O is the centre of a circle of radius 12.5 cm.

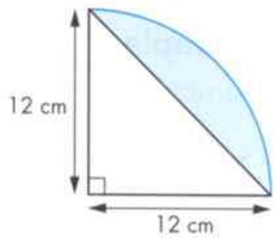
Calculate the length of the arc ACB.



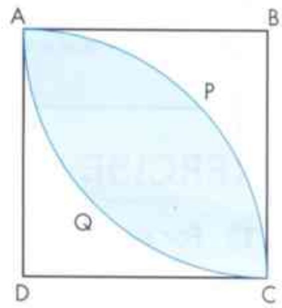
EXTENDED

EXTENDED

6 The diagram shows quarter of a circle. Calculate the area of the shaded shape, giving your answer in terms of π.



7 ABCD is a square of side length 8 cm. APC and AQC are arcs of the circles with centres D and B. Calculate the area of the shaded part, giving your answer in terms of π.

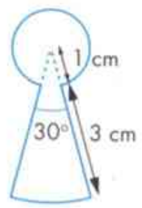


8 Find:

a the perimeter

b the area

of this shape.



27.11 Volume of a pyramid

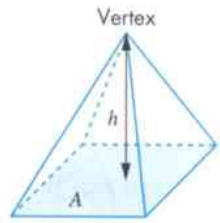
A **pyramid** is a three-dimensional shape with a base from which triangular faces rise to a common **vertex**. The base can be any polygon, but is usually a triangle, a rectangle or a square.

The volume of a pyramid is given by:

volume = $\frac{1}{3}$ × base area × **vertical height**

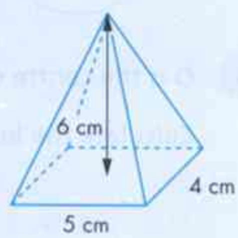
$$V = \tfrac{1}{3}Ah$$

where A is the base area and h is the vertical height.



Example 16

Calculate the volume of the pyramid.



Base area = 5 × 4 = 20 cm^2

Volume = $\frac{1}{3}$ × 20 × 6 = 40 cm^3

Example 17

A pyramid, with a square base of side 8 cm, has a volume of 320 cm³. What is the vertical height of the pyramid?

Let h be the vertical height of the pyramid. Then:

$$\text{volume} = \frac{1}{3} \times 64 \times h = 320 \text{ cm}^3$$

$$\frac{64h}{3} = 320 \text{ cm}^3$$

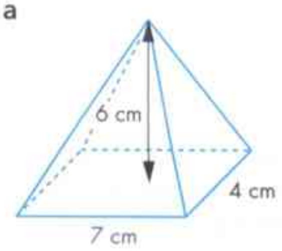
$$h = \frac{960}{64} \text{ cm}$$

$$h = 15 \text{ cm}$$

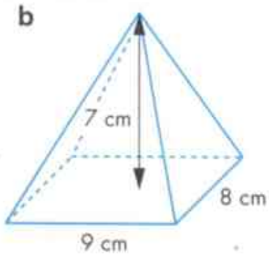
EXERCISE 27K

1. Calculate the volume of each of these pyramids, all with rectangular bases.

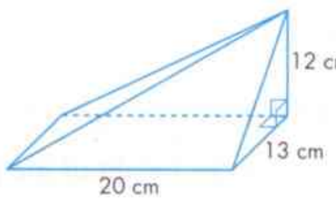
a



b



c



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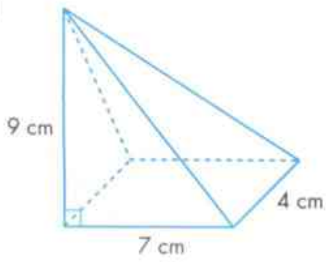

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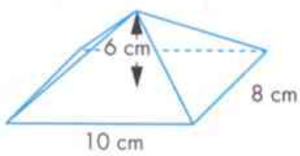
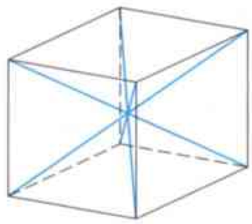


2. Calculate the volume of a pyramid that has a square base of side 9 cm and a vertical height of 10 cm.

3. Suppose you have six pyramids which have a height that is half the side of the square base.

   a Explain how they can fit together to make a cube.

   b How does this show that the formula for the volume of a pyramid is correct?



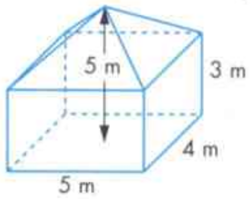
4. The glass pyramid outside the Louvre Museum in Paris was built in the 1980s. It is 20.6 m tall and the base is a square of side 35 m.

   Suppose that instead of a pyramid, the building was a cuboid with the same square base, a flat roof and the same volume. How high would it have been?

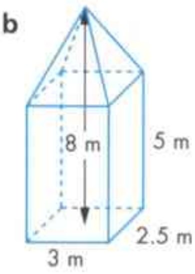
CORE

5 Calculate the volume of each of these shapes.

a



b



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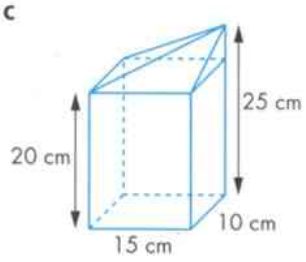
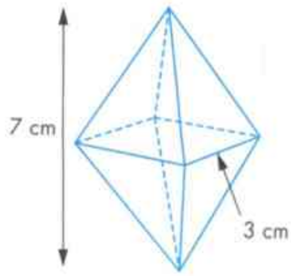


6 A crystal is in the form of two square-based pyramids joined at their bases (see diagram).

The crystal has a mass of 31.5 g.

What is the mass of 1 cm^3 of the substance?



7 A pyramid has a square base of side 6.4 cm. Its volume is 81.3 cm^3.

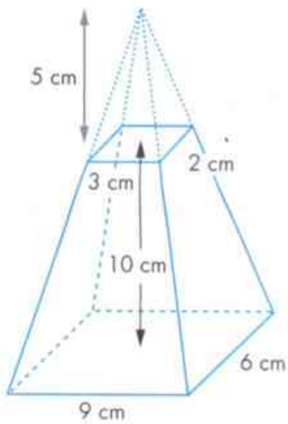
Calculate the height of the pyramid.

8 A pyramid has the same volume as a cube of side 10.0 cm.

The height of the pyramid is the same as the side of the square base.

Calculate the height of the pyramid.

9 The pyramid in the diagram has its top 5 cm cut off as shown. The shape that is left is called a frustum. Calculate the volume of the frustum.

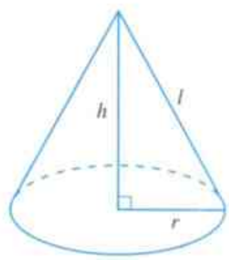


27.12 Volume and surface area of a cone

A **cone** can be treated as a pyramid with a circular base. Therefore, the formula for the volume of a cone is the same as that for a pyramid.

volume = $\frac{1}{3}$ × base area × vertical height

$$V = \frac{1}{3}\pi r^2 h$$



where r is the radius of the base and h is the **vertical height** of the cone.

The area of the curved surface of a cone is given by:

area of curved surface = π × radius × slant height

$$S = \pi r l$$

where l is the **slant height** of the cone.

So the total surface area of a cone is given by the are of the curved surface plus the area of its circular base.

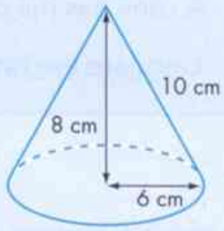
$$A = \pi r l + \pi r^2$$

Example 18

For the cone in the diagram, calculate:

a its volume **b** its total surface area.

Give your answers in terms of π.



a The volume is given by $V = \frac{1}{3}\pi r^2 h$
$= \frac{1}{3} \times \pi \times 36 \times 8 = 96\pi \text{ cm}^3$

b The total surface area is given by $A = \pi r l + \pi r^2$
$= \pi \times 6 \times 10 + \pi \times 36 = 96\pi \text{ cm}^2$

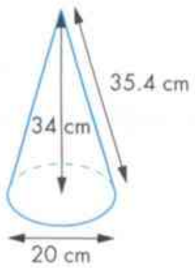
EXERCISE 27L

1 For each cone, calculate:

i its volume **ii** its total surface area.

Give your answers to 3 significant figures.

a



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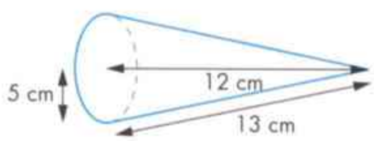

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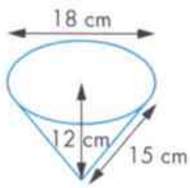
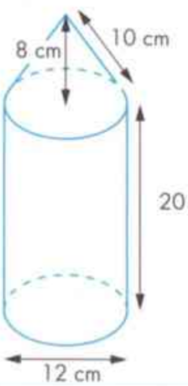


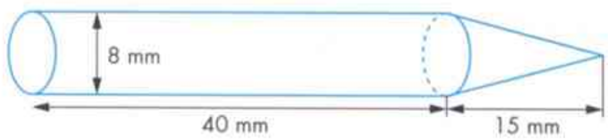
2 Find the total surface area of a cone with base radius 3 cm and slant height 5 cm. Give your answer in terms of π.

3 Calculate the volume of each of these shapes. Give your answers in terms of π.

a



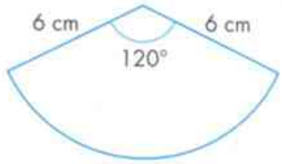
b



CORE

4 You could work with a partner on this question.

A sector of a circle, as in the diagram, can be made into a cone (without a base) by sticking the two straight edges together.



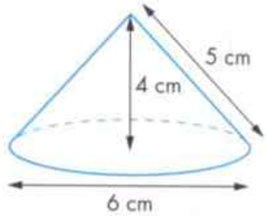
a What would be the diameter of the base of the cone in this case?

b What is the diameter if the angle is changed to 180°?

c Investigate other angles.

5 A cone has the dimensions shown in the diagram.

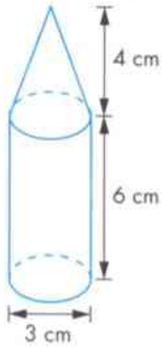
Calculate the total surface area, leaving your answer in terms of π.



6 The slant height of a cone is equal to the base diameter. Show that the area of the curved surface is twice the area of the base.

7 The model shown on the right is made from aluminium.

What is the mass of the model, given that the density of aluminium is 2.7 g/cm^3? (This means that 1 cm^3 of aluminium has a mass of 2.7 g.)



8 A container in the shape of a cone, base radius 10 cm and vertical height 19 cm, is full of water. The water is poured into an empty cylinder of radius 15 cm. How high is the water in the cylinder?

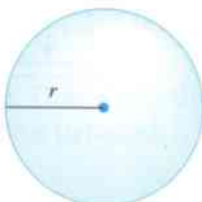
27.13 Volume and surface area of a sphere

The volume of a **sphere**, radius r, is given by:

$$V = \frac{4}{3}\pi r^3$$

Its surface area is given by:

$$A = 4\pi r^2$$



Example 19

A sphere has a radius of 8 cm. Calculate:

a its volume

b its surface area.

a The volume is given by:

$V = \frac{4}{3}\pi r^3$

$= \frac{4}{3} \times \pi \times 8^3 = \frac{2048}{3} \times \pi$

$= 2140\ \text{cm}^3$ (3 significant figures)

b The surface area is given by:

$A = 4\pi r^2$

$= 4 \times \pi \times 8^2 = 256 \times \pi$

$= 804\ \text{cm}^2$ (3 significant figures)

EXERCISE 27M

CORE

1. Calculate the volume and surface area of each of these spheres. Give your answers in terms of π.

   a Radius 3 cm   **b** Radius 6 cm   c Diameter 20 cm

2. Calculate the volume and the surface area of a sphere with a diameter of 50 cm.

3. A sphere fits exactly into an open cubical box of side 25 cm. Calculate:

   a the surface area of the sphere   **b** the volume of the sphere.

4. A metal sphere of radius 15 cm is melted down and recast into a solid cylinder of radius 6 cm. Calculate the height of the cylinder.

5. Lead has a density of 11.35 g/cm³. Calculate the maximum number of lead spheres of radius 1.5 mm that can be made from 1 kg of lead.

6. A sphere has a radius of 5.0 cm. A cone has a base radius of 8.0 cm. The sphere and the cone have the same volume. Calculate the height of the cone.

7. A sphere of diameter 10 cm is carved out of a wooden block in the shape of a cube of side 10 cm. What percentage of the wood is wasted?

Check your progress

Core

- I can carry out calculations involving the perimeter and area of a rectangle, triangle, parallelogram and trapezium
- I can calculate the areas of compound two-dimensional shapes
- I can calculate the circumference and area of a circle, leaving π in the answer if required
- I can calculate the arc length and area of sectors that are a simple fraction of a circle
- I can calculate the volume and surface area of a cuboid, prism, cylinder, sphere, pyramid and cone
- I can calculate the volumes of compound three-dimensional shapes

Extended

- I can calculate the arc length and area of sectors

Chapter 28 Symmetry

Topics	Level	Key words
1 Lines of symmetry	CORE	line of symmetry, mirror line
2 Rotational symmetry	CORE	rotational symmetry, order of rotational symmetry
3 Symmetry of special two-dimensional shapes	CORE	rectangle, square, parallelogram, kite, rhombus, trapezium, isosceles triangle, equilateral triangle
4 Symmetry of three-dimensional shapes	EXTENDED	plane of symmetry, axis of symmetry, cuboid, prism, pyramid, cone, cylinder, reflection
5 Symmetry in circles	EXTENDED	centre, chord, tangent, perpendicular bisector

In this chapter you will learn how to:

CORE

- Recognise rotational and line symmetry (including order of rotational symmetry) in two dimensions. (C4.6 and E4.6)
- Recognise properties of triangles, quadrilaterals and circles directly related to their symmetries. (C4.6 and E3.5)

EXTENDED

- Recognise symmetry properties of the prism (including cylinder) and the pyramid (including cone). (E4.6)
- Use the following symmetry properties of circles:
  - equal chords are equidistant from the centre
  - the perpendicular bisector of a chord passes through the centre
  - tangents from an external point are equal in length. (E4.6)

Why this chapter matters

If you look carefully, you will be able to spot symmetry all around you. It is present in the natural world and in objects made by humans. But, does it have a purpose and why do we need it?

Symmetry in nature, art and literature

Symmetry is everywhere you look in nature. Plants and animals have symmetrical body shapes and patterns. For example, if you divide a leaf in half, you will see that one half is the same shape as the other half.

Where is the symmetry in this butterfly, star fish and peacock? What effect does this symmetry have?

Pegasus.

This painting by a Dutch artist called M.C. Escher (1898–1972) uses line symmetry and rotational symmetry. Why do you think Escher used symmetry in his paintings?

Every tiger has its own unique pattern of stripes. These appear on the tiger's skin as well as its fur. What purpose do these symmetrical stripes serve?

Can you identify symmetry in the face of the tiger?

Symmetry in structures

St Peter's Basilica, Rome.

St Peter's Basilica, in the Vatican City in Rome, was started in 1506 and completed in 1626. It is a very symmetrical structure – see if you can identify all the symmetry that is present.

Why do you think that the designers of this building used symmetry?

These examples show some of the uses of symmetry in the world. Now think about where symmetry occurs in your own life – how important is it to you?

28.1 Lines of symmetry

Many two-dimensional shapes have one or more lines of symmetry.

A **line of symmetry** is a line that can be drawn through a shape so that what can be seen on one side of the line is the mirror image of what is on the other side. This is why a line of symmetry is sometimes called a **mirror line**.

It is also the line along which a shape can be folded exactly onto itself.

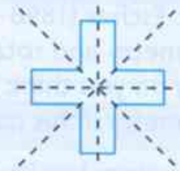
Advice and Tips

Remember you can use tracing paper to check for symmetry. For line symmetry, use it to find the mirror line. To check for rotational symmetry, trace the shape and turn your tracing around, over the shape.

Example 1

Find the number of lines of symmetry for this cross.

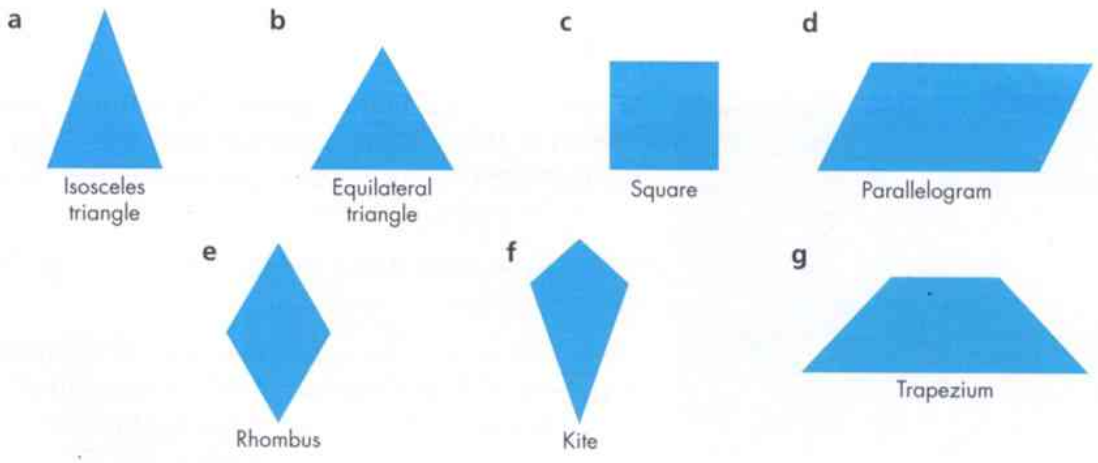
There are four altogether.



EXERCISE 28A

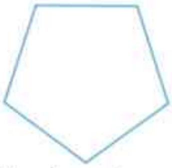
CORE

1 Copy these shapes and draw on the lines of symmetry for each one. If it will help you, use tracing paper or a mirror to check your results.



2 a Find the number of lines of symmetry for each of these regular polygons.

i



Regular pentagon

ii

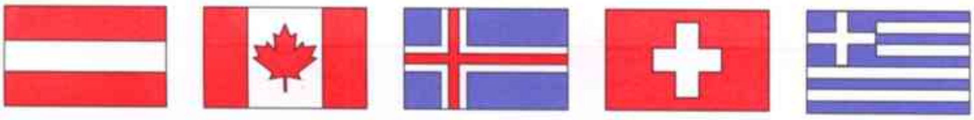
Regular hexagon

iii

Regular octagon

b How many lines of symmetry do you think a regular decagon has? (A decagon is a ten-sided polygon.)

3 Write down the number of lines of symmetry for each of these flags.



Austria Canada Iceland Switzerland Greece

4 These road signs all have lines of symmetry. Copy them and draw on the lines of symmetry for each one.

5 The animal and plant kingdoms are full of symmetry. Four examples are given below. State the number of lines of symmetry for each one.

a

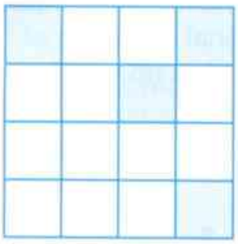
b

c

d

Can you find other examples? Find suitable pictures, copy them and state the number of lines of symmetry each one has.

6 Copy this diagram.



On your copy, shade in four more squares so that the diagram has four lines of symmetry.